

The Transcendence of π

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Abstract

The proof that π is transcendental is not well-known despite the fact that it isn't too difficult for a university mathematics student to follow. The purpose of this paper is to make the proof more widely available. A bonus is that the proof also shows that e is transcendental as well.

The material in these notes are not mine; it is taken from a supplement issued by Ian Stewart as an adjunct to a Rings and Fields course in 1970 at the University of Warwick.

Definition. A complex number is **algebraic** over \mathbb{Q} if it is a root of a polynomial equation with rational coefficients.

Thus a is algebraic if there are rational numbers $\alpha_0, \alpha_1, \dots, \alpha_n$ not all 0, such that $\alpha_0 a^n + \alpha_1 a^{n-1} + \dots + \alpha_{n-1} a + \alpha_n = 0$.

Definition. A complex number is **transcendental** if it is not algebraic, so it is not the root of any polynomial equation with rational coefficients.

In proving that it is impossible to 'square the circle' by a ruler-and-compass construction we have to appeal to the theorem:

The real number π is transcendental over \mathbb{Q}

The purpose of this supplement is to indicate, for those who may be interested, how this theorem may be proved.

It is possible to prove that there exist transcendental real numbers by using infinite cardinals, as was first done by Cantor in 1874. Earlier Liouville (1844) had actually constructed transcendentals, for example $\sum_{n=1}^{\infty} 10^{-n!}$ is transcendental.¹

¹A proof can be found at <http://rutherglen.ics.mq.edu.au/math334s106/m2334.Dioph.Liouville.pdf>

However, no naturally occurring real number (such as e or π) was proved transcendental until Hermite (1873) disposed of e . π held out until 1882 when Lindemann, using methods related to those of Hermite, disposed of that. In 1900 David Hilbert proposed the problem:

If a, b are real numbers algebraic over \mathbb{Q} , if $a \neq 0$ or 1 and b is irrational, prove a^b is transcendental.

This was solved independently in 1934 by the Russian, Gelfond, and a German, Schneider.

Before proving transcendence of π we shall prove a number of similar theorems, using simpler versions of the final method, as an aid to comprehension. The tools needed are first-year analysis.²

Theorem 1. π is irrational

Proof. Let $I_n(x) = \int_{-1}^{+1} (1-x^2)^n \cos(\alpha x) dx$

Integrating by parts we have

$$\alpha^2 I_n = 2n(2n-1)I_{n-1} - 4n(n-1)I_{n-2} \quad (n \geq 2)$$

which implies that

$$\alpha^{2n+1} I_n = n! (P_n \sin(\alpha) + Q_n \cos(\alpha)) \quad (*)$$

where P_n, Q_n are polynomials of degree $< 2n+1$ in α with integer coefficients.

Remark. $\deg P_n = n$, $\deg Q_n = n-1$

Put $\alpha = \frac{\pi}{2}$, and assume π is rational, so that $\pi = \frac{b}{a}$, $a, b \in \mathbb{Z}$

From (*) we deduce that $J_n = \frac{b^{2n+1} I_n}{n!}$ is an integer. On the other hand $J_n \rightarrow 0$ as $n \rightarrow \infty$ since b is fixed and I_n is bounded by

$$\int_{-1}^{+1} \cos\left(\frac{\pi x}{2}\right) dx$$

J_n is an integer, $\rightarrow 0$. Thus $J_n = 0$ for some n . But this integrand is continuous, and is > 0 in most of the range $(-1, +1)$, so $J_n \neq 0$. Contradiction. ■

²This was true in 1970. Is it still true today?

Theorem 2. π^2 is irrational (so π does not lie in any quadratic extension of \mathbb{Q})

Proof. Assume $\pi^2 = \frac{a}{b}$, $a, b \in \mathbb{Z}$.

Define

$$f(x) = \frac{x^n (1-x)^n}{n!},$$

$$G(x) = b^n [\pi^{2n} f(x) - \pi^{2n-2} f''(x) + \dots + (-1)^n \pi^0 f^{(2n)}(x)]$$

(superscripts indicating differentiations). We see that the value of any derivative of f at 0 or 1 is either 0 or an integer. Also $G(0)$ and $G(1)$ are integers. Now

$$\begin{aligned} \frac{d}{dx} [G'(x) \sin(\pi x) - \pi G(x) \cos(\pi x)] &= [G''(x) + \pi^2 G(x)] \sin(\pi x) \\ &= b^n \pi^{2n+2} f(x) \sin(\pi x) \quad \text{since } f^{(2n+2)}(x) = 0 \\ &= \pi^2 a^n \sin(\pi x) f(x) \end{aligned}$$

so that

$$\begin{aligned} \pi \int_0^1 a^n \sin(\pi x) f(x) dx &= \left[\frac{G'(x) \sin(\pi x)}{\pi} - G(x) \cos(\pi x) \right]_0^1 \\ &= 0 + G(0) + G(1) \\ &= \text{integer.} \end{aligned}$$

But again the integral is non-zero and $\rightarrow 0$ as $n \rightarrow \infty$. Thus again we have a contradiction. ■

Getting more involved, now:

Theorem 3 (Hermite). e is transcendental over \mathbb{Q}

Proof. Suppose $a_m e^m + \dots + a_1 e + a_0 = 0$ ($a_i \in \mathbb{Z}$). WLOG $a_0 \neq 0$

$$\text{Define } f(x) = \frac{x^{p-1}(x-1)^p(x-2)^p \dots (x-m)^p}{(p-1)!}$$

where for the moment p is arbitrary and prime.

$$\text{Define } F(x) = f(x) + f'(x) + \dots + f^{(mp+p-1)}(x).$$

Now if $0 < x < m$,

$$\begin{aligned} |f(x)| &\leq \frac{m^{p-1} m^{mp}}{(p-1)!} \\ &= \frac{m^{mp+p-1}}{(p-1)!} \end{aligned}$$

Also $\frac{d}{dx}(e^{-x}F(x)) = e^{-x}[F'(x) - F(x)] = -e^{-x}f(x)$

so that

$$\begin{aligned} a_j \int_0^j e^{-x} f(x) dx &= a_j [-e^{-x}F(x)]_0^j \\ &= a_j F(0) - a_j e^{-j} F(j). \end{aligned}$$

Multiplying by e^j and summing over $j = 0, 1, \dots, m$ we get

$$\begin{aligned} \sum_{j=0}^m a_j e^j \int_0^j e^{-x} f(x) dx &= F(0) \cdot 0 - \sum_{j=0}^m a_j F(j) \\ &= - \sum_{j=0}^m \sum_{i=0}^{mp+p-1} a_j f^{(i)}(j). \end{aligned}$$

We claim that each $f^{(i)}(j)$ is an integer, divisible by p except when $j = 0$ and $i = p - 1$. For only non-zero terms arise from terms where the factor $(x - j)^p$ has been differentiated p times, and then $p!$ cancels $(p - 1)!$ and leaves p , except in the exceptional case.

We show that in the exceptional case the term is NOT divisible by p . Clearly $f^{(p-1)}(0) = (-1)^p \dots (-m)^p$. We CHOOSE p larger than m , when this product cannot have a prime factor p . Hence the right-hand side of the above equation is an integer $\neq 0$. But as $p \rightarrow \infty$ the left-hand side tends to 0, using the above estimate for $|f(x)|$. This is a contradiction. ■

Theorem 4 (Lindemann). π is transcendental over \mathbb{Q}

Proof. If π satisfies an algebraic equation with coefficients in \mathbb{Q} , so does $i\pi$ ($i = \sqrt{-1}$). Let this equation be $\theta_1(x) = 0$, with roots $i\pi = \alpha_1, \dots, \alpha_n$. Now $e^{i\pi} + 1 = 0$ so

$$(e^{\alpha_1} + 1) \dots (e^{\alpha_n} + 1) = 0$$

We now construct an algebraic equation with integer coefficients whose roots are the exponents of e in the expansion of the above product. For example, the exponents in pairs are $\alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \dots, \alpha_{n-1} + \alpha_n$. The α s satisfy a polynomial equation over \mathbb{Q} so their elementary symmetric functions are rational. Hence the elementary symmetric functions of the sums of pairs are symmetric functions of the α s and are also rational. Thus the pairs are roots of the equation $\theta_2(x) = 0$ with rational coefficients. Similarly sums of 3 α s are roots of $\theta_3(x) = 0$, etc. Then the equation

$$\theta_1(x)\theta_2(x) \dots \theta_n(x) = 0$$

is a polynomial equation over \mathbb{Q} whose roots are all sums of α s. Deleting zero roots from this, if any, we get

$$\begin{aligned}\theta(x) &= 0 \\ \theta(x) &= cx^r + c_1x^{r-1} + \dots c_r\end{aligned}$$

and $c_r \neq 0$ since we have deleted zero roots. The roots of this equation are the non-zero exponents of e in the product when expanded. Call these β_1, \dots, β_r . The original equation becomes

$$e^{\beta_1} + \dots e^{\beta_r} + e^0 + \dots e^0 = 0$$

ie

$$\sum e^{\beta_i} + k = 0$$

where k is an integer > 0 ($\neq 0$ since the term $1 \dots 1$ exists)

Now define

$$f(x) = c^s x^{p-1} \frac{[\theta(x)]^p}{(p-1)!}$$

where $s = rp - 1$ and p will be determined later.

Define

$$\begin{aligned}F(x) &= f(x) + f'(x) + \dots + f^{(s+p)}(x). \\ \frac{d}{dx} [e^{-x} F(x)] &= -e^{-x} f(x) \text{ as before.}\end{aligned}$$

Hence we have

$$e^{-x} F(x) - F(0) = - \int_0^x e^{-y} f(y) dy$$

Putting $y = \lambda x$ we get

$$F(x) - e^x F(0) = -x \int_0^1 e^{(1-\lambda)x} f(\lambda x) d\lambda.$$

Let x range over the β_i and sum. Since $\sum e^{\beta_i} + k = 0$ we get

$$\sum_{j=1}^r F(\beta_j) + kF(0) = - \sum_{j=1}^r \beta_j \int_0^1 e^{(1-\lambda)\beta_j} f(\lambda\beta_j) d\lambda.$$

CLAIM. For large enough p the LHS is a non-zero integer.

$\sum_{j=1}^r f^{(t)}(\beta_j) = 0$ ($0 < t < p$) by definition of f . Each derivative of order p or more has a factor p and a factor c^s , since we must differentiate $[\theta(x)]^p$ enough times to get $\neq 0$. And $f^{(t)}(\beta_j)$ is a polynomial in β_j of degree at most s . The sum is

symmetric, and so is an integer provided each coefficient is divisible by c^s , which it is. (symmetric functions are polynomials in coefficients = polynomials in $\frac{c_i}{c}$ of degree $\leq s$). Thus we have

$$\sum_{j=1}^r f^{(t)}(\beta_j) = pk_t \quad t = p, \dots, p + s.$$

Thus $LHS = (\text{integer}) + kF(0)$. What is $F(0)$?

$$\begin{aligned} f^{(t)}(0) &= 0 & t = 0, \dots, p - 2. \\ f^{(p-1)}(0) &= c^s c_r^p & (c_r \neq 0) \\ f^{(t)}(0) &= p(\text{some integer}) & t = p, p + 1, \dots \end{aligned}$$

So the LHS is an integer multiple of $p + c^s c_r^p k$. This is not divisible by p if $p > k, c, c_r$. So it is a non-zero integer. But the RHS $\rightarrow 0$ as $p \rightarrow \infty$ and we get the usual contradiction. ■