Chapter 0

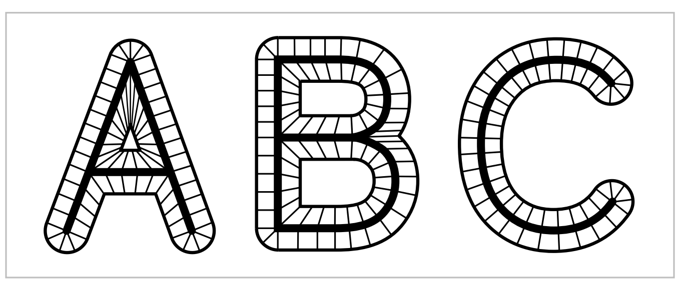
Some Underlying Geometric Notions

The aim of this short preliminary chapter is to introduce a few of the most common geometric concepts and constructions in algebraic topology. The exposition is somewhat informal, with no theorems or proofs until the last couple pages, and it should be read in this informal spirit, skipping bits here and there. In fact, this whole chapter could be skipped now, to be referred back to later for basic definitions.

To avoid overusing the word 'continuous' we adopt the convention that maps between spaces are always assumed to be continuous unless otherwise stated.

Homotopy and Homotopy Type

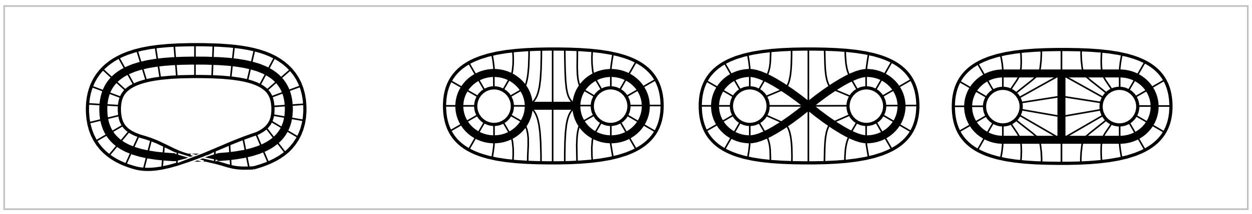
One of the main ideas of algebraic topology is to consider two spaces to be equivalent if they have 'the same shape' in a sense that is much broader than homeomorphism. To take an everyday example, the letters of the alphabet can be written either as unions of finitely many straight and curved line segments, or in thickened forms that are compact regions in the plane bounded by one or more simple closed curves. In each case the thin letter is a subspace of the thick letter, and we can continuously shrink the thick letter to the thin one. A nice way to do this is to decompose a thick letter, call it $\mathbf{X}$, into line segments connecting each point on the outer boundary of $\mathbf{X}$ to a unique point of the thin subletter X, as indicated in the figure. Then we can shrink $\mathbf{X}$ to X by sliding each point of $\mathbf{X} - X$ into X along the line segment that contains it. Points that are already in X do not move.



We can think of this shrinking process as taking place during a time interval $0 \le t \le 1$, and then it defines a family of functions $f_t : \mathbf{X} \to \mathbf{X}$ parametrized by $t \in I = [0,1]$, where $f_t(x)$ is the point to which a given point $x \in \mathbf{X}$ has moved at time t. Naturally we would like $f_t(x)$ to depend continuously on both t and x, and this will be true if we have each $x \in \mathbf{X} - \mathsf{X}$ move along its line segment at constant speed so as to reach its image point in X at time $t = 1$, while points $x \in \mathsf{X}$ are stationary, as remarked earlier.

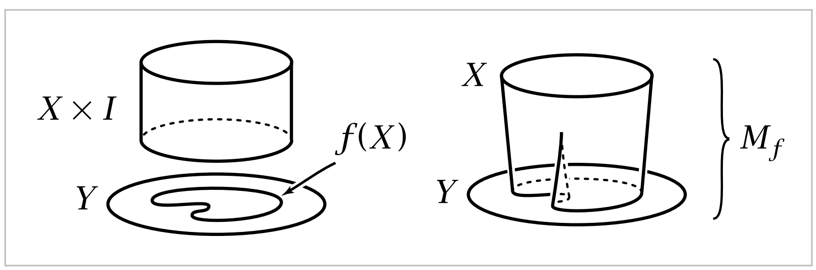
Examples of this sort lead to the following general definition. A **deformation retraction** of a space X onto a subspace A is a family of maps $f_t : X \to X$, $t \in I$, such that $f_0 = \mathbb{1}$ (the identity map), $f_1(X) = A$, and $f_t | A = \mathbb{1}$ for all t. The family f_t should be continuous in the sense that the associated map $X \times I \to X$, $(x,t) \mapsto f_t(x)$, is continuous.

It is easy to produce many more examples similar to the letter examples, with the deformation retraction f_t obtained by sliding along line segments. The figure on the left below shows such a deformation retraction of a Möbius band onto its core circle.



The three figures on the right show deformation retractions in which a disk with two smaller open subdisks removed shrinks to three different subspaces.

In all these examples the structure that gives rise to the deformation retraction can be described by means of the following definition. For a map $f : X \to Y$, the **mapping cylinder** M_f is the quotient space of the disjoint union $(X \times I) \amalg Y$ obtained by identifying each $(x,1) \in X \times I$ with $f(x) \in Y$. In the letter examples, the space X is the outer boundary of the thick letter, Y is the thin letter, and $f : X \to Y$ sends the outer endpoint of each line segment to its inner endpoint. A similar description applies to the other examples. Then it is a general fact that a mapping cylinder M_f deformation retracts to the subspace Y by sliding each point (x,t) along the segment $\{x\} \times I \subset M_f$ to the endpoint $f(x) \in Y$. Continuity of this deformation retraction is evident in the specific examples above, and for a general $f : X \to Y$ it can be verified using Proposition A.17 in the Appendix concerning the interplay between quotient spaces and product spaces.



Not all deformation retractions arise in this simple way from mapping cylinders. For example, the thick **X** deformation retracts to the thin X, which in turn deformation retracts to the point of intersection of its two crossbars. The net result is a deformation retraction of **X** onto a point, during which certain pairs of points follow paths that merge before reaching their final destination. Later in this section we will describe a considerably more complicated example, the so-called 'house with two rooms'.

A deformation retraction $f_t : X \to X$ is a special case of the general notion of a **homotopy**, which is simply any family of maps $f_t : X \to Y$, $t \in I$, such that the associated map $F : X \times I \to Y$ given by $F(x,t) = f_t(x)$ is continuous. One says that two maps $f_0, f_1 : X \to Y$ are **homotopic** if there exists a homotopy f_t connecting them, and one writes $f_0 \simeq f_1$.

In these terms, a deformation retraction of X onto a subspace A is a homotopy from the identity map of X to a **retraction** of X onto A, a map $r : X \to X$ such that $r(X) = A$ and $r | A = \mathbb{1}$. One could equally well regard a retraction as a map $X \to A$ restricting to the identity on the subspace $A \subset X$. From a more formal viewpoint a retraction is a map $r : X \to X$ with $r^2 = r$, since this equation says exactly that r is the identity on its image. Retractions are the topological analogs of projection operators in other parts of mathematics.

Not all retractions come from deformation retractions. For example, a space X always retracts onto any point $x_0 \in X$ via the constant map sending all of X to x_0, but a space that deformation retracts onto a point must be path-connected since a deformation retraction of X to x_0 gives a path joining each $x \in X$ to x_0. It is less trivial to show that there are path-connected spaces that do not deformation retract onto a point. One would expect this to be the case for the letters 'with holes', A, B, D, O, P, Q, R. In Chapter 1 we will develop techniques to prove this.

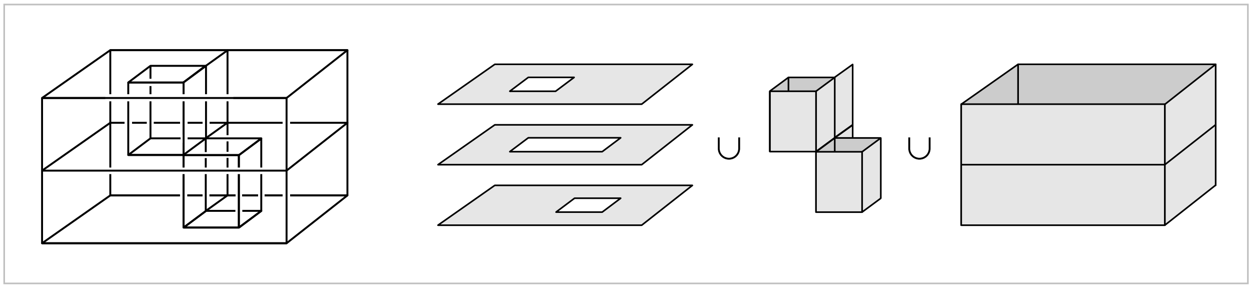
A homotopy $f_t : X \to X$ that gives a deformation retraction of X onto a subspace A has the property that $f_t | A = \mathbb{1}$ for all t. In general, a homotopy $f_t : X \to Y$ whose restriction to a subspace $A \subset X$ is independent of t is called a **homotopy relative to** A, or more concisely, a homotopy rel A. Thus, a deformation retraction of X onto A is a homotopy rel A from the identity map of X to a retraction of X onto A.

If a space X deformation retracts onto a subspace A via $f_t : X \to X$, then if $r : X \to A$ denotes the resulting retraction and $i : A \to X$ the inclusion, we have $ri = \mathbb{1}$ and $ir \simeq \mathbb{1}$, the latter homotopy being given by f_t. Generalizing this situation, a map $f : X \to Y$ is called a **homotopy equivalence** if there is a map $g : Y \to X$ such that $fg \simeq \mathbb{1}$ and $gf \simeq \mathbb{1}$. The spaces X and Y are said to be **homotopy equivalent** or to have the same **homotopy type**. The notation is $X \simeq Y$. It is an easy exercise to check that this is an equivalence relation, in contrast with the nonsymmetric notion of deformation retraction. For example, the three graphs  are all homotopy equivalent since they are deformation retracts of the same space, as we saw earlier, but none of the three is a deformation retract of any other.

It is true in general that two spaces X and Y are homotopy equivalent if and only if there exists a third space Z containing both X and Y as deformation retracts. For the less trivial implication one can in fact take Z to be the mapping cylinder M_f of any homotopy equivalence $f : X \to Y$. We observed previously that M_f deformation retracts to Y, so what needs to be proved is that M_f also deformation retracts to its other end X if f is a homotopy equivalence. This is shown in Corollary 0.21.

A space having the homotopy type of a point is called **contractible**. This amounts to requiring that the identity map of the space be **nullhomotopic**, that is, homotopic to a constant map. In general, this is slightly weaker than saying the space deformation retracts to a point; see the exercises at the end of the chapter for an example distinguishing these two notions.

Let us describe now an example of a 2-dimensional subspace of $\mathbb{R}^3$, known as the *house with two rooms*, which is contractible but not in any obvious way. To build this



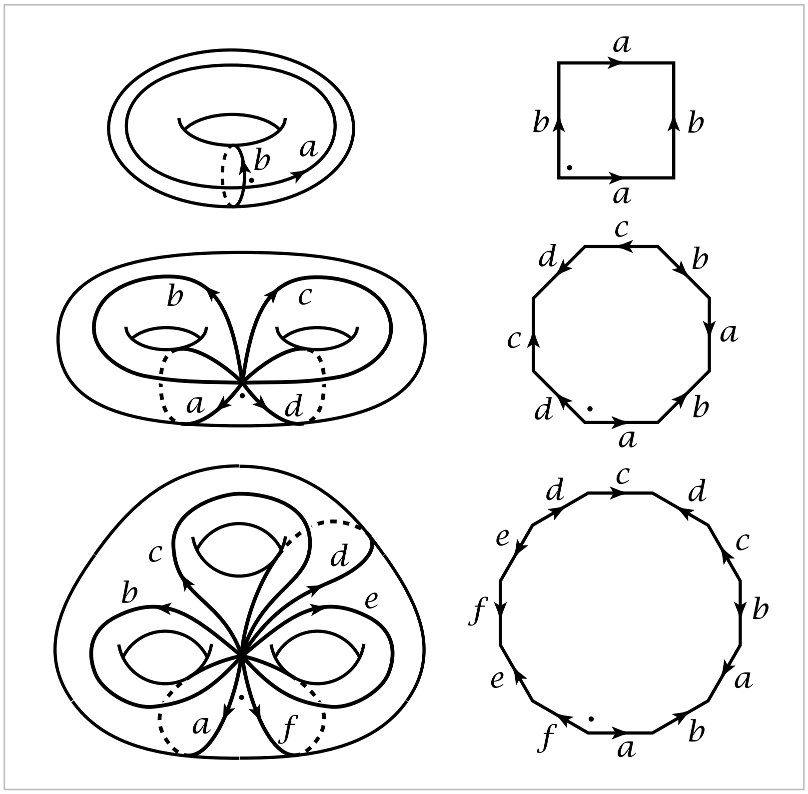
space, start with a box divided into two chambers by a horizontal rectangle, where by a 'rectangle' we mean not just the four edges of a rectangle but also its interior. Access to the two chambers from outside the box is provided by two vertical tunnels. The upper tunnel is made by punching out a square from the top of the box and another square directly below it from the middle horizontal rectangle, then inserting four vertical rectangles, the walls of the tunnel. This tunnel allows entry to the lower chamber from outside the box. The lower tunnel is formed in similar fashion, providing entry to the upper chamber. Finally, two vertical rectangles are inserted to form 'support walls' for the two tunnels. The resulting space X thus consists of three horizontal pieces homeomorphic to annuli plus all the vertical rectangles that form the walls of the two chambers.

To see that X is contractible, consider a closed ε-neighborhood $N(X)$ of X. This clearly deformation retracts onto X if ε is sufficiently small. In fact, $N(X)$ is the mapping cylinder of a map from the boundary surface of $N(X)$ to X. Less obvious is the fact that $N(X)$ is homeomorphic to D^3, the unit ball in $\mathbb{R}^3$. To see this, imagine forming $N(X)$ from a ball of clay by pushing a finger into the ball to create the upper tunnel, then gradually hollowing out the lower chamber, and similarly pushing a finger in to create the lower tunnel and hollowing out the upper chamber. Mathematically, this process gives a family of embeddings $h_t : D^3 \to \mathbb{R}^3$ starting with the usual inclusion $D^3 \hookrightarrow \mathbb{R}^3$ and ending with a homeomorphism onto $N(X)$.

Thus we have $X \simeq N(X) = D^3 \simeq point$, so X is contractible since homotopy equivalence is an equivalence relation. In fact, X deformation retracts to a point. For if f_t is a deformation retraction of the ball $N(X)$ to a point $x_0 \in X$ and if $r : N(X) \to X$ is a retraction, for example the end result of a deformation retraction of $N(X)$ to X, then the restriction of the composition rf_t to X is a deformation retraction of X to x_0. However, it is quite a challenging exercise to see exactly what this deformation retraction looks like.

Cell Complexes

A familiar way of constructing the torus $S^1 \times S^1$ is by identifying opposite sides of a square. More generally, an orientable surface M_g of genus g can be constructed from a polygon with $4g$ sides by identifying pairs of edges, as shown in the figure in the first three cases $g = 1, 2, 3$. The $4g$ edges of the polygon become a union of $2g$ circles in the surface, all intersecting in a single point. The interior of the polygon can be thought of as an open disk, or a **2-cell**, attached to the union of the $2g$ circles. One can also regard the union of the circles as being obtained from their common point of intersection, by attaching $2g$ open arcs, or **1-cells**. Thus the surface can be built up in stages: Start with a point, attach 1-cells to this point, then attach a 2-cell.



A natural generalization of this is to construct a space by the following procedure:

(1) Start with a discrete set X^0, whose points are regarded as 0-cells.

(2) Inductively, form the ***n*-skeleton** X^n from X^{n-1} by attaching n-cells e^n_α via maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$. This means that X^n is the quotient space of the disjoint union $X^{n-1} \coprod_\alpha D^n_\alpha$ of X^{n-1} with a collection of n-disks D^n_α under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$. Thus as a set, $X^n = X^{n-1} \coprod_\alpha e^n_\alpha$ where each e^n_α is an open n-disk.

(3) One can either stop this inductive process at a finite stage, setting $X = X^n$ for some $n < \infty$, or one can continue indefinitely, setting $X = \bigcup_n X^n$. In the latter case X is given the weak topology: A set $A \subset X$ is open (or closed) iff $A \cap X^n$ is open (or closed) in X^n for each n.

A space X constructed in this way is called a **cell complex** or **CW complex.** The explanation of the letters 'CW' is given in the Appendix, where a number of basic topological properties of cell complexes are proved. The reader who wonders about various point-set topological questions lurking in the background of the following discussion should consult the Appendix for details.

If $X = X^n$ for some n, then X is said to be finite-dimensional, and the smallest such n is the **dimension** of X, the maximum dimension of cells of X.

Example 0.1. A 1-dimensional cell complex $X = X^1$ is what is called a **graph** in algebraic topology. It consists of vertices (the 0-cells) to which edges (the 1-cells) are attached. The two ends of an edge can be attached to the same vertex.

Example 0.2. The house with two rooms, pictured earlier, has a visually obvious 2-dimensional cell complex structure. The 0-cells are the vertices where three or more of the depicted edges meet, and the 1-cells are the interiors of the edges connecting these vertices. This gives the 1-skeleton X^1, and the 2-cells are the components of the remainder of the space, $X - X^1$. If one counts up, one finds there are 29 0-cells, 51 1-cells, and 23 2-cells, with the alternating sum $29 - 51 + 23$ equal to 1. This is the **Euler characteristic**, which for a cell complex with finitely many cells is defined to be the number of even-dimensional cells minus the number of odd-dimensional cells. As we shall show in Theorem 2.44, the Euler characteristic of a cell complex depends only on its homotopy type, so the fact that the house with two rooms has the homotopy type of a point implies that its Euler characteristic must be 1, no matter how it is represented as a cell complex.

Example 0.3. The sphere S^n has the structure of a cell complex with just two cells, e^0 and e^n, the n-cell being attached by the constant map $S^{n-1} \to e^0$. This is equivalent to regarding S^n as the quotient space $D^n/\partial D^n$.

Example 0.4. **Real projective *n*-space** $\mathbb{R}P^n$ is defined to be the space of all lines through the origin in $\mathbb{R}^{n+1}$. Each such line is determined by a nonzero vector in $\mathbb{R}^{n+1}$, unique up to scalar multiplication, and $\mathbb{R}P^n$ is topologized as the quotient space of $\mathbb{R}^{n+1} - \{0\}$ under the equivalence relation $v \sim \lambda v$ for scalars $\lambda \neq 0$. We can restrict to vectors of length 1, so $\mathbb{R}P^n$ is also the quotient space $S^n/(v \sim -v)$, the sphere with antipodal points identified. This is equivalent to saying that $\mathbb{R}P^n$ is the quotient space of a hemisphere D^n with antipodal points of ∂D^n identified. Since ∂D^n with antipodal points identified is just $\mathbb{R}P^{n-1}$, we see that $\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$ by attaching an n-cell, with the quotient projection $S^{n-1} \to \mathbb{R}P^{n-1}$ as the attaching map. It follows by induction on n that $\mathbb{R}P^n$ has a cell complex structure $e^0 \cup e^1 \cup \cdots \cup e^n$ with one cell e^i in each dimension $i \leq n$.

Example 0.5. Since $\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$ by attaching an n-cell, the infinite union $\mathbb{R}P^\infty = \bigcup_n \mathbb{R}P^n$ becomes a cell complex with one cell in each dimension. We can view $\mathbb{R}P^\infty$ as the space of lines through the origin in $\mathbb{R}^\infty = \bigcup_n \mathbb{R}^n$.

Example 0.6. **Complex projective *n*-space** $\mathbb{C}P^n$ is the space of complex lines through the origin in $\mathbb{C}^{n+1}$, that is, 1-dimensional vector subspaces of $\mathbb{C}^{n+1}$. As in the case of $\mathbb{R}P^n$, each line is determined by a nonzero vector in $\mathbb{C}^{n+1}$, unique up to scalar multiplication, and $\mathbb{C}P^n$ is topologized as the quotient space of $\mathbb{C}^{n+1} - \{0\}$ under the equivalence relation $v \sim \lambda v$ for $\lambda \neq 0$. Equivalently, this is the quotient of the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ with $v \sim \lambda v$ for $|\lambda| = 1$. It is also possible to obtain $\mathbb{C}\mathrm{P}^n$ as a quotient space of the disk D^{2n} under the identifications $v \sim \lambda v$ for $v \in \partial D^{2n}$, in the following way. The vectors in $S^{2n+1} \subset \mathbb{C}^{n+1}$ with last coordinate real and nonnegative are precisely the vectors of the form $(w, \sqrt{1-|w|^2}\,) \in \mathbb{C}^n \times \mathbb{C}$ with $|w| \le 1$. Such vectors form the graph of the function $w \mapsto \sqrt{1-|w|^2}$. This is a disk D^{2n}_+ bounded by the sphere $S^{2n-1} \subset S^{2n+1}$ consisting of vectors $(w,0) \in \mathbb{C}^n \times \mathbb{C}$ with $|w| = 1$. Each vector in S^{2n+1} is equivalent under the identifications $v \sim \lambda v$ to a vector in D^{2n}_+, and the latter vector is unique if its last coordinate is nonzero. If the last coordinate is zero, we have just the identifications $v \sim \lambda v$ for $v \in S^{2n-1}$.

From this description of $\mathbb{C}\mathrm{P}^n$ as the quotient of D^{2n}_+ under the identifications $v \sim \lambda v$ for $v \in S^{2n-1}$ it follows that $\mathbb{C}\mathrm{P}^n$ is obtained from $\mathbb{C}\mathrm{P}^{n-1}$ by attaching a cell e^{2n} via the quotient map $S^{2n-1} \to \mathbb{C}\mathrm{P}^{n-1}$. So by induction on n we obtain a cell structure $\mathbb{C}\mathrm{P}^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$ with cells only in even dimensions. Similarly, $\mathbb{C}\mathrm{P}^\infty$ has a cell structure with one cell in each even dimension.

After these examples we return now to general theory. Each cell e^n_α in a cell complex X has a **characteristic map** $\Phi_\alpha : D^n_\alpha \to X$ which extends the attaching map φ_α and is a homeomorphism from the interior of D^n_α onto e^n_α. Namely, we can take Φ_α to be the composition $D^n_\alpha \hookrightarrow X^{n-1} \coprod_\alpha D^n_\alpha \to X^n \hookrightarrow X$ where the middle map is the quotient map defining X^n. For example, in the canonical cell structure on S^n described in Example 0.3, a characteristic map for the n-cell is the quotient map $D^n \to S^n$ collapsing ∂D^n to a point. For $\mathbb{R}\mathrm{P}^n$ a characteristic map for the cell e^i is the quotient map $D^i \to \mathbb{R}\mathrm{P}^i \subset \mathbb{R}\mathrm{P}^n$ identifying antipodal points of ∂D^i, and similarly for $\mathbb{C}\mathrm{P}^n$.

A **subcomplex** of a cell complex X is a closed subspace $A \subset X$ that is a union of cells of X. Since A is closed, the characteristic map of each cell in A has image contained in A, and in particular the image of the attaching map of each cell in A is contained in A, so A is a cell complex in its own right. A pair (X, A) consisting of a cell complex X and a subcomplex A will be called a **CW pair**.

For example, each skeleton X^n of a cell complex X is a subcomplex. Particular cases of this are the subcomplexes $\mathbb{R}\mathrm{P}^k \subset \mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^k \subset \mathbb{C}\mathrm{P}^n$ for $k \le n$. These are in fact the only subcomplexes of $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$.

There are natural inclusions $S^0 \subset S^1 \subset \cdots \subset S^n$, but these subspheres are not subcomplexes of S^n in its usual cell structure with just two cells. However, we can give S^n a different cell structure in which each of the subspheres S^k is a subcomplex, by regarding each S^k as being obtained inductively from the equatorial S^{k-1} by attaching two k-cells, the components of $S^k - S^{k-1}$. The infinite-dimensional sphere $S^\infty = \bigcup_n S^n$ then becomes a cell complex as well. Note that the two-to-one quotient map $S^\infty \to \mathbb{R}\mathrm{P}^\infty$ that identifies antipodal points of S^∞ identifies the two n-cells of S^∞ to the single n-cell of $\mathbb{R}\mathrm{P}^\infty$.

In the examples of cell complexes given so far, the closure of each cell is a subcomplex, and more generally the closure of any collection of cells is a subcomplex. Most naturally arising cell structures have this property, but it need not hold in general. For example, if we start with S^1 with its minimal cell structure and attach to this a 2-cell by a map $S^1 \to S^1$ whose image is a nontrivial subarc of S^1, then the closure of the 2-cell is not a subcomplex since it contains only a part of the 1-cell.

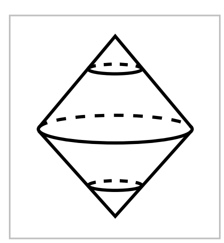
Operations on Spaces

Cell complexes have a very nice mixture of rigidity and flexibility, with enough rigidity to allow many arguments to proceed in a combinatorial cell-by-cell fashion and enough flexibility to allow many natural constructions to be performed on them. Here are some of those constructions.

Products. If X and Y are cell complexes, then $X \times Y$ has the structure of a cell complex with cells the products $e^m_\alpha \times e^n_\beta$ where e^m_α ranges over the cells of X and e^n_β ranges over the cells of Y. For example, the cell structure on the torus $S^1 \times S^1$ described at the beginning of this section is obtained in this way from the standard cell structure on S^1. For completely general CW complexes X and Y there is one small complication: The topology on $X \times Y$ as a cell complex is sometimes finer than the product topology, with more open sets than the product topology has, though the two topologies coincide if either X or Y has only finitely many cells, or if both X and Y have countably many cells. This is explained in the Appendix. In practice this subtle issue of point-set topology rarely causes problems, however.

Quotients. If (X, A) is a CW pair consisting of a cell complex X and a subcomplex A, then the quotient space X/A inherits a natural cell complex structure from X. The cells of X/A are the cells of $X - A$ plus one new 0-cell, the image of A in X/A. For a cell e^n_α of $X - A$ attached by $\varphi_\alpha : S^{n-1} \to X^{n-1}$, the attaching map for the corresponding cell in X/A is the composition $S^{n-1} \to X^{n-1} \to X^{n-1}/A^{n-1}$.

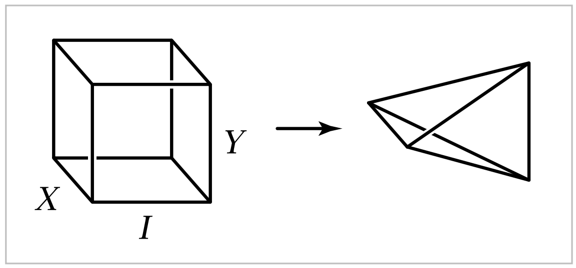
For example, if we give S^{n-1} any cell structure and build D^n from S^{n-1} by attaching an n-cell, then the quotient D^n/S^{n-1} is S^n with its usual cell structure. As another example, take X to be a closed orientable surface with the cell structure described at the beginning of this section, with a single 2-cell, and let A be the complement of this 2-cell, the 1-skeleton of X. Then X/A has a cell structure consisting of a 0-cell with a 2-cell attached, and there is only one way to attach a cell to a 0-cell, by the constant map, so X/A is S^2.



Suspension. For a space X, the **suspension** SX is the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point. The motivating example is $X = S^n$, when $SX = S^{n+1}$ with the two 'suspension points' at the north and south poles of S^{n+1}, the points $(0, \cdots, 0, \pm 1)$. One can regard SX as a double cone on X, the union of two copies of the **cone** $CX = (X\times I)/(X\times\{0\})$. If X is a CW complex, so are SX and CX as quotients of $X\times I$ with its product cell structure, I being given the standard cell structure of two 0-cells joined by a 1-cell.

Suspension becomes increasingly important the farther one goes into algebraic topology, though why this should be so is certainly not evident in advance. One especially useful property of suspension is that not only spaces but also maps can be suspended. Namely, a map $f:X\to Y$ suspends to $Sf:SX\to SY$, the quotient map of $f\times \mathbb{1}:X\times I\to Y\times I$.

Join. The cone CX is the union of all line segments joining points of X to an external vertex, and similarly the suspension SX is the union of all line segments joining points of X to two external vertices. More generally, given X and a second space Y, one can define the space of all line segments joining points in X to points in Y. This is the **join** $X*Y$, the quotient space of $X\times Y\times I$ under the identifications $(x,y_1,0)\sim(x,y_2,0)$ and $(x_1,y,1)\sim(x_2,y,1)$. Thus we are collapsing the subspace $X\times Y\times\{0\}$ to X and $X\times Y\times\{1\}$ to Y. For example, if X and Y are both closed intervals, then we are collapsing two opposite faces of a cube onto line segments so that the cube becomes a tetrahedron. In the general case, $X*Y$ contains copies of X and Y at its two ends, and every other point (x,y,t) in $X*Y$ is on a unique line segment joining the point $x\in X\subset X*Y$ to the point $y\in Y\subset X*Y$, the segment obtained by fixing x and y and letting the coordinate t in (x,y,t) vary.



A nice way to write points of $X*Y$ is as formal linear combinations t_1x+t_2y with $0\le t_i\le 1$ and $t_1+t_2=1$, subject to the rules $0x+1y=y$ and $1x+0y=x$ that correspond exactly to the identifications defining $X*Y$. In much the same way, an iterated join $X_1*\cdots*X_n$ can be constructed as the space of formal linear combinations $t_1x_1+\cdots+t_nx_n$ with $0\le t_i\le 1$ and $t_1+\cdots+t_n=1$, with the convention that terms $0x_i$ can be omitted. A very special case that plays a central role in algebraic topology is when each X_i is just a point. For example, the join of two points is a line segment, the join of three points is a triangle, and the join of four points is a tetrahedron. In general, the join of n points is a convex polyhedron of dimension $n-1$ called a **simplex**. Concretely, if the n points are the n standard basis vectors for $\mathbb{R}^n$, then their join is the $(n-1)$-dimensional simplex

$$\Delta^{n-1}=\{\,(t_1,\cdots,t_n)\in\mathbb{R}^n \mid t_1+\cdots+t_n=1 \text{ and } t_i\ge 0\,\}$$

Another interesting example is when each X_i is S^0, two points. If we take the two points of X_i to be the two unit vectors along the i^{th} coordinate axis in $\mathbb{R}^n$, then the join $X_1*\cdots*X_n$ is the union of 2^n copies of the simplex Δ^{n-1}, and radial projection from the origin gives a homeomorphism between $X_1*\cdots*X_n$ and S^{n-1}.

If X and Y are CW complexes, then there is a natural CW structure on $X * Y$ having the subspaces X and Y as subcomplexes, with the remaining cells being the product cells of $X \times Y \times (0,1)$. As usual with products, the CW topology on $X * Y$ may be finer than the quotient of the product topology on $X \times Y \times I$.

Wedge Sum. This is a rather trivial but still quite useful operation. Given spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$, then the **wedge sum** $X \vee Y$ is the quotient of the disjoint union $X \amalg Y$ obtained by identifying x_0 and y_0 to a single point. For example, $S^1 \vee S^1$ is homeomorphic to the figure '8', two circles touching at a point. More generally one could form the wedge sum $\bigvee_\alpha X_\alpha$ of an arbitrary collection of spaces X_α by starting with the disjoint union $\coprod_\alpha X_\alpha$ and identifying points $x_\alpha \in X_\alpha$ to a single point. In case the spaces X_α are cell complexes and the points x_α are 0-cells, then $\bigvee_\alpha X_\alpha$ is a cell complex since it is obtained from the cell complex $\coprod_\alpha X_\alpha$ by collapsing a subcomplex to a point.

For any cell complex X, the quotient X^n/X^{n-1} is a wedge sum of n-spheres $\bigvee_\alpha S^n_\alpha$, with one sphere for each n-cell of X.

Smash Product. Like suspension, this is another construction whose importance becomes evident only later. Inside a product space $X \times Y$ there are copies of X and Y, namely $X \times \{y_0\}$ and $\{x_0\} \times Y$ for points $x_0 \in X$ and $y_0 \in Y$. These two copies of X and Y in $X \times Y$ intersect only at the point (x_0, y_0), so their union can be identified with the wedge sum $X \vee Y$. The **smash product** $X \wedge Y$ is then defined to be the quotient $X \times Y / X \vee Y$. One can think of $X \wedge Y$ as a reduced version of $X \times Y$ obtained by collapsing away the parts that are not genuinely a product, the separate factors X and Y.

The smash product $X \wedge Y$ is a cell complex if X and Y are cell complexes with x_0 and y_0 0-cells, assuming that we give $X \times Y$ the cell-complex topology rather than the product topology in cases when these two topologies differ. For example, $S^m \wedge S^n$ has a cell structure with just two cells, of dimensions 0 and $m+n$, hence $S^m \wedge S^n = S^{m+n}$. In particular, when $m = n = 1$ we see that collapsing longitude and meridian circles of a torus to a point produces a 2-sphere.

Two Criteria for Homotopy Equivalence

Earlier in this chapter the main tool we used for constructing homotopy equivalences was the fact that a mapping cylinder deformation retracts onto its 'target' end. By repeated application of this fact one can often produce homotopy equivalences between rather different-looking spaces. However, this process can be a bit cumbersome in practice, so it is useful to have other techniques available as well. We will describe two commonly used methods here. The first involves collapsing certain subspaces to points, and the second involves varying the way in which the parts of a space are put together.

Collapsing Subspaces

The operation of collapsing a subspace to a point usually has a drastic effect on homotopy type, but one might hope that if the subspace being collapsed already has the homotopy type of a point, then collapsing it to a point might not change the homotopy type of the whole space. Here is a positive result in this direction:

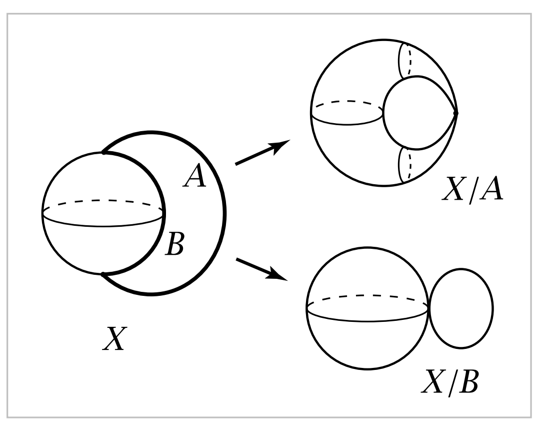
If (X, A) is a CW pair consisting of a CW complex X and a contractible subcomplex A, then the quotient map $X \to X/A$ is a homotopy equivalence.

A proof will be given later in Proposition 0.17, but for now let us look at some examples showing how this result can be applied.

Example 0.7: Graphs. The three graphs    are homotopy equivalent since each is a deformation retract of a disk with two holes, but we can also deduce this from the collapsing criterion above since collapsing the middle edge of the first and third graphs produces the second graph.

More generally, suppose X is any graph with finitely many vertices and edges. If the two endpoints of any edge of X are distinct, we can collapse this edge to a point, producing a homotopy equivalent graph with one fewer edge. This simplification can be repeated until all edges of X are loops, and then each component of X is either an isolated vertex or a wedge sum of circles.

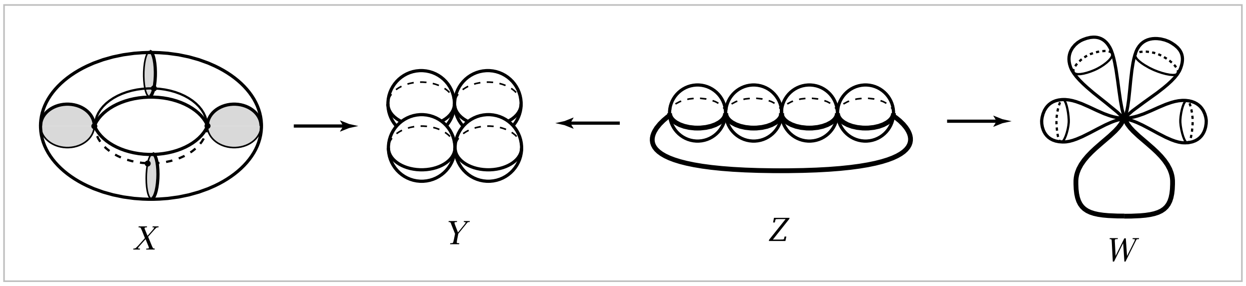
This raises the question of whether two such graphs, having only one vertex in each component, can be homotopy equivalent if they are not in fact just isomorphic graphs. Exercise 12 at the end of the chapter reduces the question to the case of connected graphs. Then the task is to prove that a wedge sum $\bigvee_m S^1$ of m circles is not homotopy equivalent to $\bigvee_n S^1$ if $m \neq n$. This sort of thing is hard to do directly. What one would like is some sort of algebraic object associated to spaces, depending only on their homotopy type, and taking different values for $\bigvee_m S^1$ and $\bigvee_n S^1$ if $m \neq n$. In fact the Euler characteristic does this since $\bigvee_m S^1$ has Euler characteristic $1-m$. But it is a rather nontrivial theorem that the Euler characteristic of a space depends only on its homotopy type. A different algebraic invariant that works equally well for graphs, and whose rigorous development requires less effort than the Euler characteristic, is the fundamental group of a space, the subject of Chapter 1.



Example 0.8. Consider the space X obtained from S^2 by attaching the two ends of an arc A to two distinct points on the sphere, say the north and south poles. Let B be an arc in S^2 joining the two points where A attaches. Then X can be given a CW complex structure with the two endpoints of A and B as 0-cells, the interiors of A and B as 1-cells, and the rest of S^2 as a 2-cell. Since A and B are contractible,

X/A and X/B are homotopy equivalent to X. The space X/A is the quotient S^2/S^0, the sphere with two points identified, and X/B is $S^1 \vee S^2$. Hence S^2/S^0 and $S^1 \vee S^2$ are homotopy equivalent, a fact which may not be entirely obvious at first glance.

Example 0.9. Let X be the union of a torus with n meridional disks. To obtain a CW structure on X, choose a longitudinal circle in the torus, intersecting each of the meridional disks in one point. These intersection points are then the 0-cells, the 1-cells are the rest of the longitudinal circle and the boundary circles of the meridional disks, and the 2-cells are the remaining regions of the torus and the interiors of the meridional disks. Collapsing each meridional disk to a point yields a homotopy



equivalent space Y consisting of n 2-spheres, each tangent to its two neighbors, a 'necklace with n beads'. The third space Z in the figure, a strand of n beads with a string joining its two ends, collapses to Y by collapsing the string to a point, so this collapse is a homotopy equivalence. Finally, by collapsing the arc in Z formed by the front halves of the equators of the n beads, we obtain the fourth space W, a wedge sum of S^1 with n 2-spheres. (One can see why a wedge sum is sometimes called a 'bouquet' in the older literature.)

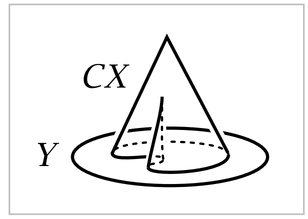
Example 0.10: Reduced Suspension. Let X be a CW complex and $x_0 \in X$ a 0-cell. Inside the suspension SX we have the line segment $\{x_0\}\times I$, and collapsing this to a point yields a space ΣX homotopy equivalent to SX, called the **reduced suspension** of X. For example, if we take X to be $S^1 \vee S^1$ with x_0 the intersection point of the two circles, then the ordinary suspension SX is the union of two spheres intersecting along the arc $\{x_0\}\times I$, so the reduced suspension ΣX is $S^2 \vee S^2$, a slightly simpler space. More generally we have $\Sigma(X \vee Y) = \Sigma X \vee \Sigma Y$ for arbitrary CW complexes X and Y. Another way in which the reduced suspension ΣX is slightly simpler than SX is in its CW structure. In SX there are two 0-cells (the two suspension points) and an $(n+1)$-cell $e^n \times (0,1)$ for each n-cell e^n of X, whereas in ΣX there is a single 0-cell and an $(n+1)$-cell for each n-cell of X other than the 0-cell x_0.

The reduced suspension ΣX is actually the same as the smash product $X \wedge S^1$ since both spaces are the quotient of $X \times I$ with $X \times \partial I \cup \{x_0\}\times I$ collapsed to a point.

Attaching Spaces

Another common way to change a space without changing its homotopy type involves the idea of continuously varying how its parts are attached together. A general definition of 'attaching one space to another' that includes the case of attaching cells is the following. We start with a space X_0 and another space X_1 that we wish to attach to X_0 by identifying the points in a subspace $A \subset X_1$ with points of X_0. The data needed to do this is a map $f : A \to X_0$, for then we can form a quotient space of $X_0 \amalg X_1$ by identifying each point $a \in A$ with its image $f(a) \in X_0$. Let us denote this quotient space by $X_0 \sqcup_f X_1$, the space X_0 with X_1 **attached along** A **via** f. When $(X_1, A) = (D^n, S^{n-1})$ we have the case of attaching an n-cell to X_0 via a map $f : S^{n-1} \to X_0$.

Mapping cylinders are examples of this construction, since the mapping cylinder M_f of a map $f : X \to Y$ is the space obtained from Y by attaching $X \times I$ along $X \times \{1\}$ via f. Closely related to the mapping cylinder M_f is the **mapping cone** $C_f = Y \sqcup_f CX$ where CX is the cone $(X \times I)/(X \times \{0\})$ and we attach this to Y along $X \times \{1\}$ via the identifications $(x, 1) \sim f(x)$. For example, when X is a sphere S^{n-1} the mapping cone C_f is the space obtained from Y by attaching an n-cell via $f : S^{n-1} \to Y$. A mapping cone C_f can also be viewed as the quotient M_f / X of the mapping cylinder M_f with the subspace $X = X \times \{0\}$ collapsed to a point.

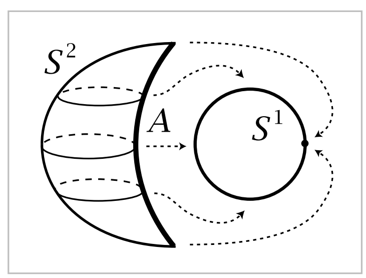


If one varies an attaching map f by a homotopy f_t, one gets a family of spaces whose shape is undergoing a continuous change, it would seem, and one might expect these spaces all to have the same homotopy type. This is often the case:

If (X_1, A) is a CW pair and the two attaching maps $f, g : A \to X_0$ are homotopic, then $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$.

Again let us defer the proof and look at some examples.

Example 0.11. Let us rederive the result in Example 0.8 that a sphere with two points identified is homotopy equivalent to $S^1 \vee S^2$. The sphere with two points identified can be obtained by attaching S^2 to S^1 by a map that wraps a closed arc A in S^2 around S^1, as shown in the figure. Since A is contractible, this attaching map is homotopic to a constant map, and attaching S^2 to S^1 via a constant map of A yields $S^1 \vee S^2$. The result then follows since (S^2, A) is a CW pair, S^2 being obtained from A by attaching a 2-cell.



Example 0.12. In similar fashion we can see that the necklace in Example 0.9 is homotopy equivalent to the wedge sum of a circle with n 2-spheres. The necklace can be obtained from a circle by attaching n 2-spheres along arcs, so the necklace is homotopy equivalent to the space obtained by attaching n 2-spheres to a circle at points. Then we can slide these attaching points around the circle until they all coincide, producing the wedge sum.

Example 0.13. Here is an application of the earlier fact that collapsing a contractible subcomplex is a homotopy equivalence: If (X, A) is a CW pair, consisting of a cell complex X and a subcomplex A, then $X/A \simeq X \cup CA$, the mapping cone of the inclusion $A \hookrightarrow X$. For we have $X/A = (X \cup CA)/CA \simeq X \cup CA$ since CA is a contractible subcomplex of $X \cup CA$.

Example 0.14. If (X, A) is a CW pair and A is contractible in X, that is, the inclusion $A \hookrightarrow X$ is homotopic to a constant map, then $X/A \simeq X \vee SA$. Namely, by the previous example we have $X/A \simeq X \cup CA$, and then since A is contractible in X, the mapping cone $X \cup CA$ of the inclusion $A \hookrightarrow X$ is homotopy equivalent to the mapping cone of a constant map, which is $X \vee SA$. For example, $S^n/S^i \simeq S^n \vee S^{i+1}$ for $i < n$, since S^i is contractible in S^n if $i < n$. In particular this gives $S^2/S^0 \simeq S^2 \vee S^1$, which is Example 0.8 again.

The Homotopy Extension Property

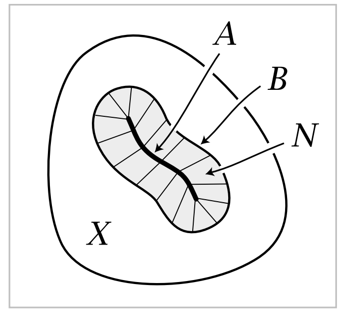
In this final section of the chapter we will actually prove a few things, including the two criteria for homotopy equivalence described above. The proofs depend upon a technical property that arises in many other contexts as well. Consider the following problem. Suppose one is given a map $f_0 : X \to Y$, and on a subspace $A \subset X$ one is also given a homotopy $f_t : A \to Y$ of $f_0 | A$ that one would like to extend to a homotopy $f_t : X \to Y$ of the given f_0. If the pair (X, A) is such that this extension problem can always be solved, one says that (X, A) has the **homotopy extension property**. Thus (X, A) has the homotopy extension property if every pair of maps $X \times \{0\} \to Y$ and $A \times I \to Y$ that agree on $A \times \{0\}$ can be extended to a map $X \times I \to Y$.

A pair (X, A) has the homotopy extension property if and only if $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.

For one implication, the homotopy extension property for (X, A) implies that the identity map $X \times \{0\} \cup A \times I \to X \times \{0\} \cup A \times I$ extends to a map $X \times I \to X \times \{0\} \cup A \times I$, so $X \times \{0\} \cup A \times I$ is a retract of $X \times I$. The converse is equally easy when A is closed in X. Then any two maps $X \times \{0\} \to Y$ and $A \times I \to Y$ that agree on $A \times \{0\}$ combine to give a map $X \times \{0\} \cup A \times I \to Y$ which is continuous since it is continuous on the closed sets $X \times \{0\}$ and $A \times I$. By composing this map $X \times \{0\} \cup A \times I \to Y$ with a retraction $X \times I \to X \times \{0\} \cup A \times I$ we get an extension $X \times I \to Y$, so (X, A) has the homotopy extension property. The hypothesis that A is closed can be avoided by a more complicated argument given in the Appendix. If $X \times \{0\} \cup A \times I$ is a retract of $X \times I$ and X is Hausdorff, then A must in fact be closed in X. For if $r : X \times I \to X \times I$ is a retraction onto $X \times \{0\} \cup A \times I$, then the image of r is the set of points $z \in X \times I$ with $r(z) = z$, a closed set if X is Hausdorff, so $X \times \{0\} \cup A \times I$ is closed in $X \times I$ and hence A is closed in X.

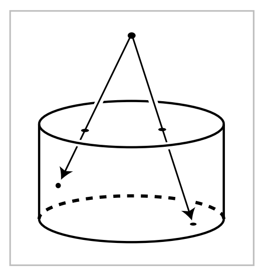
A simple example of a pair (X, A) with A closed for which the homotopy extension property fails is the pair (I, A) where $A = \{0, 1, 1/2, 1/3, 1/4, \cdots\}$. It is not hard to show that there is no continuous retraction $I \times I \to I \times \{0\} \cup A \times I$. The breakdown of homotopy extension here can be attributed to the bad structure of (X, A) near 0. With nicer local structure the homotopy extension property does hold, as the next example shows.

Example 0.15. A pair (X, A) has the homotopy extension property if A has a mapping cylinder neighborhood in X, by which we mean a closed neighborhood N containing a subspace B, thought of as the boundary of N, with $N - B$ an open neighborhood of A, such that there exists a map $f : B \to A$ and a homeomorphism $h : M_f \to N$ with $h | A \cup B = 1\!\!1$. Mapping cylinder neighborhoods like this occur fairly often. For example, the thick letters discussed at the beginning of the chapter provide such neighborhoods of the thin letters, regarded as subspaces of the plane. To verify the homotopy extension property, notice first that $I \times I$ retracts onto $I \times \{0\} \cup \partial I \times I$, hence $B \times I \times I$ retracts onto $B \times I \times \{0\} \cup B \times \partial I \times I$, and this retraction induces a retraction of $M_f \times I$ onto $M_f \times \{0\} \cup (A \cup B) \times I$. Thus $(M_f, A \cup B)$ has the homotopy extension property. Hence so does the homeomorphic pair $(N, A \cup B)$. Now given a map $X \to Y$ and a homotopy of its restriction to A, we can take the constant homotopy on $X - (N - B)$ and then extend over N by applying the homotopy extension property for $(N, A \cup B)$ to the given homotopy on A and the constant homotopy on B.



Proposition 0.16. *If (X, A) is a CW pair, then $X \times \{0\} \cup A \times I$ is a deformation retract of $X \times I$, hence (X, A) has the homotopy extension property.*

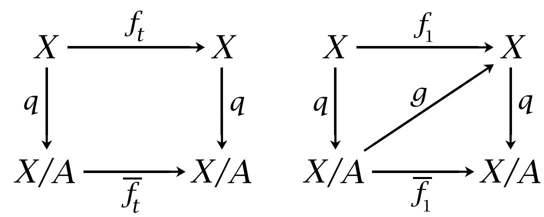
Proof: There is a retraction $r : D^n \times I \to D^n \times \{0\} \cup \partial D^n \times I$, for example the radial projection from the point $(0, 2) \in D^n \times \mathbb{R}$. Then setting $r_t = tr + (1 - t)1\!\!1$ gives a deformation retraction of $D^n \times I$ onto $D^n \times \{0\} \cup \partial D^n \times I$. This deformation retraction gives rise to a deformation retraction of $X^n \times I$ onto $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$ since $X^n \times I$ is obtained from $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$ by attaching copies of $D^n \times I$ along $D^n \times \{0\} \cup \partial D^n \times I$. If we perform the deformation retraction of $X^n \times I$ onto $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$ during the t-interval $[1/2^{n+1}, 1/2^n]$, this infinite concatenation of homotopies is a deformation retraction of $X \times I$ onto $X \times \{0\} \cup A \times I$. There is no problem with continuity of this deformation retraction at $t = 0$ since it is continuous on $X^n \times I$, being stationary there during the t-interval $[0, 1/2^{n+1}]$, and CW complexes have the weak topology with respect to their skeleta so a map is continuous iff its restriction to each skeleton is continuous. □



Now we can prove a generalization of the earlier assertion that collapsing a contractible subcomplex is a homotopy equivalence.

Proposition 0.17. *If the pair (X, A) satisfies the homotopy extension property and A is contractible, then the quotient map $q : X \to X/A$ is a homotopy equivalence.*

Proof: Let $f_t : X \to X$ be a homotopy extending a contraction of A, with $f_0 = \mathbb{1}$. Since $f_t(A) \subset A$ for all t, the composition $qf_t : X \to X/A$ sends A to a point and hence factors as a composition $X \xrightarrow{q} X/A \to X/A$. Denoting the latter map by $\overline{f}_t : X/A \to X/A$, we have $qf_t = \overline{f}_t q$ in the first of the two diagrams at the right. When $t = 1$ we have $f_1(A)$ equal to a point, the point to which A contracts, so f_1 induces a map $g : X/A \to X$ with $gq = f_1$, as in the second diagram. It follows that $qg = \overline{f}_1$ since $qg(\overline{x}) = qgq(x) = qf_1(x) = \overline{f}_1 q(x) = \overline{f}_1(\overline{x})$. The maps g and q are inverse homotopy equivalences since $gq = f_1 \simeq f_0 = \mathbb{1}$ via f_t and $qg = \overline{f}_1 \simeq \overline{f}_0 = \mathbb{1}$ via $\overline{f}_t$. □



Another application of the homotopy extension property, giving a slightly more refined version of one of our earlier criteria for homotopy equivalence, is the following:

Proposition 0.18. *If (X_1, A) is a CW pair and we have attaching maps $f, g : A \to X_0$ that are homotopic, then $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$ rel X_0.*

Here the definition of $W \simeq Z$ rel Y for pairs (W, Y) and (Z, Y) is that there are maps $\varphi : W \to Z$ and $\psi : Z \to W$ restricting to the identity on Y, such that $\psi\varphi \simeq \mathbb{1}$ and $\varphi\psi \simeq \mathbb{1}$ via homotopies that restrict to the identity on Y at all times.

Proof: If $F : A \times I \to X_0$ is a homotopy from f to g, consider the space $X_0 \sqcup_F (X_1 \times I)$. This contains both $X_0 \sqcup_f X_1$ and $X_0 \sqcup_g X_1$ as subspaces. A deformation retraction of $X_1 \times I$ onto $X_1 \times \{0\} \cup A \times I$ as in Proposition 0.16 induces a deformation retraction of $X_0 \sqcup_F (X_1 \times I)$ onto $X_0 \sqcup_f X_1$. Similarly $X_0 \sqcup_F (X_1 \times I)$ deformation retracts onto $X_0 \sqcup_g X_1$. Both these deformation retractions restrict to the identity on X_0, so together they give a homotopy equivalence $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$ rel X_0. □

We finish this chapter with a technical result whose proof will involve several applications of the homotopy extension property:

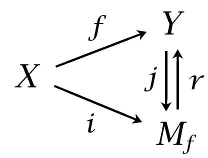
Proposition 0.19. *Suppose (X, A) and (Y, A) satisfy the homotopy extension property, and $f : X \to Y$ is a homotopy equivalence with $f | A = \mathbb{1}$. Then f is a homotopy equivalence* rel A.

Corollary 0.20. *If (X, A) satisfies the homotopy extension property and the inclusion $A \hookrightarrow X$ is a homotopy equivalence, then A is a deformation retract of X.*

Proof: Apply the proposition to the inclusion $A \hookrightarrow X$. □

Corollary 0.21. *A map $f : X \to Y$ is a homotopy equivalence iff X is a deformation retract of the mapping cylinder M_f. Hence, two spaces X and Y are homotopy equivalent iff there is a third space containing both X and Y as deformation retracts.*

Proof: In the diagram at the right the maps i and j are the inclusions and r is the canonical retraction, so $f = ri$ and $i \simeq jf$. Since j and r are homotopy equivalences, it follows that f is a homotopy equivalence iff i is a homotopy equivalence, since the composition of two homotopy equivalences is a homotopy equivalence and a map homotopic to a homotopy equivalence is a homotopy equivalence. Now apply the preceding corollary to the pair (M_f, X), which satisfies the homotopy extension property by Example 0.15 using the neighborhood $X\times[0, 1/2]$ of X in M_f. ☐



Proof of 0.19: Let $g : Y \to X$ be a homotopy inverse for f. There will be three steps to the proof:

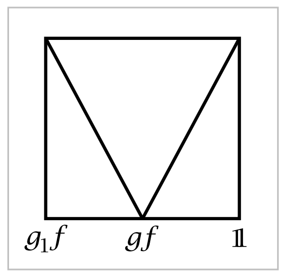
(1) Construct a homotopy from g to a map g_1 with $g_1 | A = \mathbb{1}$.
(2) Show $g_1 f \simeq \mathbb{1}$ rel A.
(3) Show $f g_1 \simeq \mathbb{1}$ rel A.

(1) Let $h_t : X \to X$ be a homotopy from $gf = h_0$ to $\mathbb{1} = h_1$. Since $f | A = \mathbb{1}$, we can view $h_t | A$ as a homotopy from $g | A$ to $\mathbb{1}$. Then since we assume (Y, A) has the homotopy extension property, we can extend this homotopy to a homotopy $g_t : Y \to X$ from $g = g_0$ to a map g_1 with $g_1 | A = \mathbb{1}$.

(2) A homotopy from $g_1 f$ to $\mathbb{1}$ is given by the formulas

$$k_t = \begin{cases} g_{1-2t} f, & 0 \le t \le 1/2 \\ h_{2t-1}, & 1/2 \le t \le 1 \end{cases}$$

Note that the two definitions agree when $t = 1/2$. Since $f | A = \mathbb{1}$ and $g_t = h_t$ on A, the homotopy $k_t | A$ starts and ends with the identity, and its second half simply retraces its first half, that is, $k_t = k_{1-t}$ on A. We will define a 'homotopy of homotopies' $k_{tu} : A \to X$ by means of the figure at the right showing the parameter domain $I\times I$ for the pairs (t, u), with the t-axis horizontal and the u-axis vertical. On the bottom edge of the square we define $k_{t0} = k_t | A$. Below the 'V' we define k_{tu} to be independent of u, and above the 'V' we define k_{tu} to be independent of t. This is unambiguous since $k_t = k_{1-t}$ on A. Since $k_0 = \mathbb{1}$ on A, we have $k_{tu} = \mathbb{1}$ for (t, u) in the left, right, and top edges of the square. Next we extend k_{tu} over X, as follows. Since (X, A) has the homotopy extension property, so does $(X\times I, A\times I)$, as one can see from the equivalent retraction property. Viewing k_{tu} as a homotopy of $k_t | A$, we can therefore extend $k_{tu} : A \to X$ to $k_{tu} : X \to X$ with $k_{t0} = k_t$. If we restrict this k_{tu} to the left, top, and right edges of the (t, u)-square, we get a homotopy $g_1 f \simeq \mathbb{1}$ rel A.



(3) Since $g_1 \simeq g$, we have $f g_1 \simeq f g \simeq \mathbb{1}$, so $f g_1 \simeq \mathbb{1}$ and steps (1) and (2) can be repeated with the pair f, g replaced by g_1, f. The result is a map $f_1 : X \to Y$ with $f_1 | A = \mathbb{1}$ and $f_1 g_1 \simeq \mathbb{1}$ rel A. Hence $f_1 \simeq f_1(g_1 f) = (f_1 g_1) f \simeq f$ rel A. From this we deduce that $f g_1 \simeq f_1 g_1 \simeq \mathbb{1}$ rel A. ☐

Exercises

1. Construct an explicit deformation retraction of the torus with one point deleted onto a graph consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus.

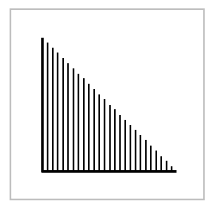
2. Construct an explicit deformation retraction of $\mathbb{R}^n - \{0\}$ onto S^{n-1}.

3. (a) Show that the composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Deduce that homotopy equivalence is an equivalence relation.
(b) Show that the relation of homotopy among maps $X \to Y$ is an equivalence relation.
(c) Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

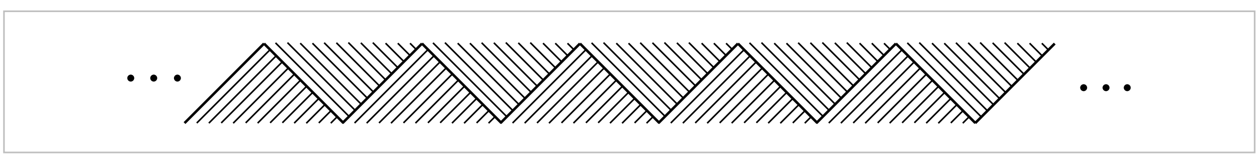
4. A **deformation retraction in the weak sense** of a space X to a subspace A is a homotopy $f_t : X \to X$ such that $f_0 = \mathbb{1}$, $f_1(X) \subset A$, and $f_t(A) \subset A$ for all t. Show that if X deformation retracts to A in this weak sense, then the inclusion $A \hookrightarrow X$ is a homotopy equivalence.

5. Show that if a space X deformation retracts to a point $x \in X$, then for each neighborhood U of x in X there exists a neighborhood $V \subset U$ of x such that the inclusion map $V \hookrightarrow U$ is nullhomotopic.

6. (a) Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0,1] \times \{0\}$ together with all the vertical segments $\{r\} \times [0, 1-r]$ for r a rational number in $[0,1]$. Show that X deformation retracts to any point in the segment $[0,1] \times \{0\}$, but not to any other point. [See the preceding problem.]

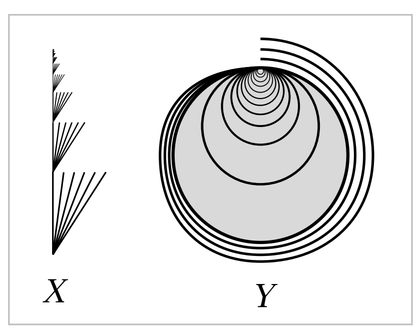


(b) Let Y be the subspace of $\mathbb{R}^2$ that is the union of an infinite number of copies of X arranged as in the figure below. Show that Y is contractible but does not deformation retract onto any point.



(c) Let Z be the zigzag subspace of Y homeomorphic to $\mathbb{R}$ indicated by the heavier line. Show there is a deformation retraction in the weak sense (see Exercise 4) of Y onto Z, but no true deformation retraction.

7. Fill in the details in the following construction from [Edwards 1999] of a compact space $Y \subset \mathbb{R}^3$ with the same properties as the space Y in Exercise 6, that is, Y is contractible but does not deformation retract to any point. To begin, let X be the union of an infinite sequence of cones on the Cantor set arranged end-to-end, as in the figure. Next, form the one-point compactification of $X \times \mathbb{R}$. This embeds in $\mathbb{R}^3$ as a closed disk with curved 'fins' attached along



circular arcs, and with the one-point compactification of X as a cross-sectional slice. The desired space Y is then obtained from this subspace of $\mathbb{R}^3$ by wrapping one more cone on the Cantor set around the boundary of the disk.

8. For $n > 2$, construct an n-room analog of the house with two rooms.

9. Show that a retract of a contractible space is contractible.

10. Show that a space X is contractible iff every map $f:X\to Y$, for arbitrary Y, is nullhomotopic. Similarly, show X is contractible iff every map $f:Y\to X$ is nullhomotopic.

11. Show that $f:X\to Y$ is a homotopy equivalence if there exist maps $g,h:Y\to X$ such that $fg \simeq \mathbb{1}$ and $hf \simeq \mathbb{1}$. More generally, show that f is a homotopy equivalence if fg and hf are homotopy equivalences.

12. Show that a homotopy equivalence $f:X\to Y$ induces a bijection between the set of path-components of X and the set of path-components of Y, and that f restricts to a homotopy equivalence from each path-component of X to the corresponding path-component of Y. Prove also the corresponding statements with components instead of path-components. Deduce that if the components of a space X coincide with its path-components, then the same holds for any space Y homotopy equivalent to X.

13. Show that any two deformation retractions r_t^0 and r_t^1 of a space X onto a subspace A can be joined by a continuous family of deformation retractions r_t^s, $0 \le s \le 1$, of X onto A, where continuity means that the map $X\times I\times I\to X$ sending (x,s,t) to $r_t^s(x)$ is continuous.

14. Given positive integers v, e, and f satisfying $v - e + f = 2$, construct a cell structure on S^2 having v 0-cells, e 1-cells, and f 2-cells.

15. Enumerate all the subcomplexes of S^∞, with the cell structure on S^∞ that has S^n as its n-skeleton.

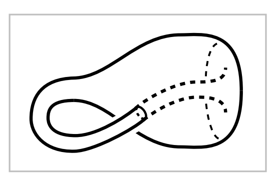
16. Show that S^∞ is contractible.

17. (a) Show that the mapping cylinder of every map $f:S^1\to S^1$ is a CW complex.
(b) Construct a 2-dimensional CW complex that contains both an annulus $S^1\times I$ and a Möbius band as deformation retracts.

18. Show that $S^1 * S^1 = S^3$, and more generally $S^m * S^n = S^{m+n+1}$.

19. Show that the space obtained from S^2 by attaching n 2-cells along any collection of n circles in S^2 is homotopy equivalent to the wedge sum of $n+1$ 2-spheres.

20. Show that the subspace $X \subset \mathbb{R}^3$ formed by a Klein bottle intersecting itself in a circle, as shown in the figure, is homotopy equivalent to $S^1 \vee S^1 \vee S^2$.



21. If X is a connected Hausdorff space that is a union of a finite number of 2-spheres, any two of which intersect in at most one point, show that X is homotopy equivalent to a wedge sum of S^1's and S^2's.

22. Let X be a finite graph lying in a half-plane $P \subset \mathbb{R}^3$ and intersecting the edge of P in a subset of the vertices of X. Describe the homotopy type of the 'surface of revolution' obtained by rotating X about the edge line of P.

23. Show that a CW complex is contractible if it is the union of two contractible subcomplexes whose intersection is also contractible.

24. Let X and Y be CW complexes with 0-cells x_0 and y_0. Show that the quotient spaces $X*Y/(X*\{y_0\}\cup\{x_0\}*Y)$ and $S(X\wedge Y)/S(\{x_0\}\wedge\{y_0\})$ are homeomorphic, and deduce that $X*Y \simeq S(X\wedge Y)$.

25. If X is a CW complex with components X_α, show that the suspension SX is homotopy equivalent to $Y\bigvee_\alpha SX_\alpha$ for some graph Y. In the case that X is a finite graph, show that SX is homotopy equivalent to a wedge sum of circles and 2-spheres.

26. Use Corollary 0.20 to show that if (X,A) has the homotopy extension property, then $X\times I$ deformation retracts to $X\times\{0\}\cup A\times I$. Deduce from this that Proposition 0.18 holds more generally for any pair (X_1,A) satisfying the homotopy extension property.

27. Given a pair (X,A) and a homotopy equivalence $f:A\to B$, show that the natural map $X\to B\sqcup_f X$ is a homotopy equivalence if (X,A) satisfies the homotopy extension property. [Hint: Consider $X\cup M_f$ and use the preceding problem.] An interesting case is when f is a quotient map, hence the map $X\to B\sqcup_f X$ is the quotient map identifying each set $f^{-1}(b)$ to a point. When B is a point this gives another proof of Proposition 0.17.

28. Show that if (X_1,A) satisfies the homotopy extension property, then so does every pair $(X_0\sqcup_f X_1, X_0)$ obtained by attaching X_1 to a space X_0 via a map $f:A\to X_0$.

29. In case the CW complex X is obtained from a subcomplex A by attaching a single cell e^n, describe exactly what the extension of a homotopy $f_t:A\to Y$ to X given by the proof of Proposition 0.16 looks like. That is, for a point $x\in e^n$, describe the path $f_t(x)$ for the extended f_t.