FROM DOMINOES TO HEXAGONS

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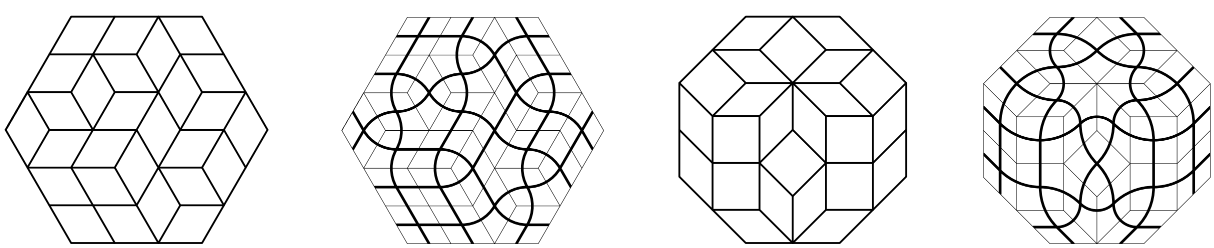
Abstract. There is a natural generalization of domino tilings to tilings of a polygon by hexagons, or, dually, configurations of oriented curves that meet in triples. We show exactly when two such tilings can be connected by a series of moves analogous to the domino flip move. The triple diagrams that result have connections to Legendrian knots, cluster algebras, and planar algebras.

1. Introduction

The study of tilings of a planar region by lozenges, as in Figure 1(a), or more generally by rhombuses, as in Figure 1(c), has a long history in combinatorics. Interesting questions include deciding when a region can be tiled, connectivity of the space of tilings under the basic move $\leftrightarrow$, counting the number of tilings, and behavior of a random tiling.

One useful tool for studying these tilings is the dual picture, as in Figures 1(b). It consists of replacing each rhombus in the tiling by a cross of two strands connecting opposite, parallel sides. If we trace a strand through the tiling it comes out on a parallel face on the opposite side of the region, independently of the tiling of the region. Lozenge tilings give patterns of strands with three families of parallel (non-intersecting strands).

If we drop the restriction to three families of parallel strands, as in Figure 1(d), and take any set of strands connecting boundary points which are both *generic* (only two strands intersect at a time) and *minimal* (no strand intersects itself and no two strands intersect more than once), then there is a corresponding dual tiling by rhombuses, as in Figure 1(c). (This was proved in great generality by Kenyon and Schlenker [KS05].)



(a) Lozenge tiling (b) Lozenge dual (c) Rhombus tiling (d) Rhombus dual

Figure 1. Tilings by lozenges or more general rhombuses and their duals.

These rhombus tilings correspond to words in the symmetric group, and any two rhombus tilings of a region can be related by Reidemeister III moves on the strands diagram, which in terms of rhombus tilings replaces a hexagon tiled with 3 rhombus tiles with another tiling of the same hexagon by the same 3 rhombi. This has been proved by several people, including by Curtis, Ingerman, and Morrow in their work on resistor networks [CIM98].

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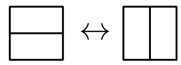
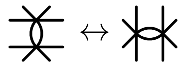
In this paper, we will look at a related situation, in which tilings by dominoes (Figure 2(a)) are given a dual picture (Figure 2(b)), in which we replace each domino by an "asterisk" of three strands which are not generic, but rather meet in a triple point. When we perform the basic *domino flip*



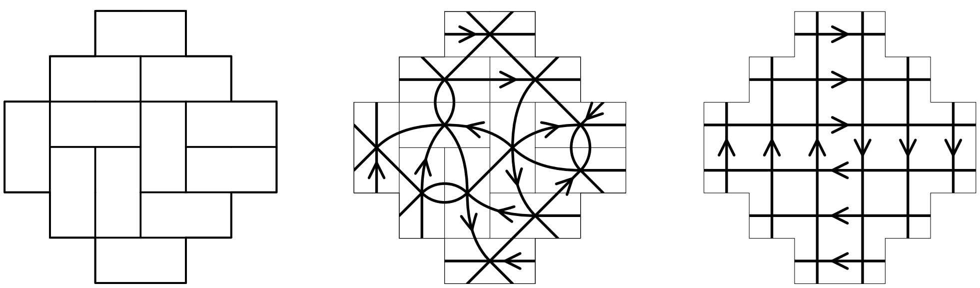
on domino tilings, the dual strands perform what we will call a $2 \leftrightarrow 2$ *move*



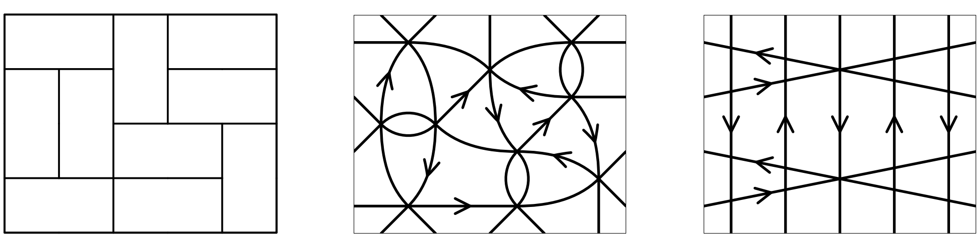
In particular, the connectivity of the strands as we trace them across the diagram is unchanged. Since the space of domino tilings is connected under the domino flip [Thu90], the connectivity of the strands depends only on the shape of the region and not on the particular tiling (Figure 2(c)).

Conversely, it follows from results in this paper that this is a faithful representation: every set of immersed arcs with the same connectivity as a domino tiling, meeting only in triple points, and with a minimal number of triple points corresponds uniquely to a domino tiling.

As in the case of rhombus tilings, strands always connect parallel sides. Unlike in the rhombus case, strands may intersect each other arbitrarily many times. For the special case of the tiling of the Aztec diamond (Figures 2(a)–2(c)), strands connecting parallel sides do not cross in the net connectivity across the diagram. But this is not true in general, as you can see in Figures 2(d)–2(f).



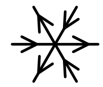
(a) An Aztec diamond (b) Aztec dual (c) Aztec connectivity



(d) A 5×4 rectangle (e) Rectangle dual (f) Rect. connectivity

FIGURE 2. Domino tilings and their duals, along with the connectivity of the strands

But there is something we can say about the crossings of strands. Consider the complementary regions to the strands. Because they meet with six around a vertex, they may be checkerboard colored with two colors, black and white. Orient the strand segments clockwise around the black regions and counterclockwise around the white regions. Because 6 is congruent to 2 modulo 4, the orientations are consistent when we follow a strand through a triple crossing:

 .

See Figure 2(e) for the orientations of the strands in the dual to the domino tiling in Figure 2(d). Another way to characterize these orientations is to consider a generic curve inside the diagram; the strands that it crosses will alternate crossing left-to-right and right-to-left.

The appropriate analogue of the minimal condition for the diagrams dual to rhombus tilings is that strands connecting parallel sides oriented in the same direction never cross each other.

These pictures suggest a natural generalization.

Definition 1. A *triple diagram* is a collection of oriented one-manifolds, possibly with boundary, mapped smoothly into the disk. The image of a connected component is a *strand*; it is either an *arc* (the image of an interval) or a *loop* (the image of a circle). The maps are required to satisfy:

- Three strands cross at each point of intersection;
- the endpoints of arcs are distinct points on the boundary of the disk, and no other points map to the boundary; and
- the orientations on the strands induce consistent orientations on the complementary regions.

Triple diagrams are considered up to homotopy among such diagrams. This makes them essentially combinatorial objects, 6-valent graphs with some extra structure.

Definition 2. A *connected* triple diagram D is a diagram in which the image of the immersed curves together with the boundary of the disk is connected. Equivalently, it is a diagram in which each complementary region to the image is a disk. A *disjoint component* of a triple diagram is a connected component of the image of the immersed curves which does not meet the boundary of the disk. A *simple loop* is a loop which goes through no simple crossings and bounds an empty disk.

A diagram in which all strands are arcs is automatically connected.

If the diagram is connected, the condition on orientations amounts to requiring that the orientations alternate in and out around each triple point, and is automatically satisfied if we start anywhere and start assigning alternating orientations around vertices. Since all the complementary regions of a connected diagram are disks, we can construct the dual, which is a topological tiling by hexagons. Note that a domino tiling can be turned into a tiling by hexagons by adding a vertex in the middle of the long edges of each domino. The triple crossing diagram described above is dual to this tiling by hexagons.

As before, the connectivity of the arcs gives a matching of the vertices on the boundary of the disk. A natural question to ask is which matchings of the boundary vertices are achievable by a triple diagram. There is one immediate restriction. The orientations on the strands alternate in and out as we go around the boundary of the disk, and every inward-pointing end gets matched with an outward-pointing end. The matching is a bijection between the in- and out-endpoints.

Theorem 3. *In a disk with $2n$ endpoints on the boundary, all $n!$ pairings of in-endpoints with out-endpoints are achievable by some triple point diagram without closed strands.*

The proof is in Section 2.

If we think about these diagrams by analogy with group theory, this theorem assures us that triple crossings *generate* admissible pairings. The next question is *relations*: What moves on diagrams of triple points allow us to pass between two diagrams which induce the same matching? One natural move is the $2 \leftrightarrow 2$ move above. This move suffices if the total number of triple points is as small as possible.

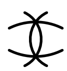
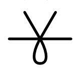
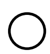
Definition 4. A *minimal* triple diagram is a connected diagram with no more triple points than any other triple diagram inducing the same pairing on the boundary.

(The condition of connectivity merely rules out small loops with no crossings disconnected from the rest of the diagram.)

Theorem 5. *Any two minimal triple point diagrams with the same matching on the endpoints can be related by a sequence of* $2 \leftrightarrow 2$ *moves.*

The restriction to minimal diagrams is necessary, since the $2 \leftrightarrow 2$ move, as per its name, keeps the total number of triple points unchanged. To drop this restriction, we need to introduce some moves that change the number of triple points while preserving the matching. These moves have no analogues in domino tilings, since the number of triple crossings in a domino tiling is the number of dominoes, i.e., half the area of the region, which is invariant.

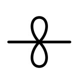
Theorem 6. *Any triple point diagram can be reduced to any minimal triple point diagram (with the same matching) by a sequence of the following moves:*

- $2 \leftrightarrow 2$ *moves*  $\leftrightarrow$ *;*
- $1 \to 0$ *moves*  $\to$ *; and*
- *Dropping a simple loop*  *with no crossings and an empty interior.*

Note that this version of the theorem is a little stronger than you might expect: we never increase the number of triple crossings.

Theorems 5 and 6 are proved in section 3. The result for domino tilings was previously known [Thu90, STCR95].

Remark. There is an alternate version of Theorem 6 which stays within the space of connected diagrams: any connected diagram can be reduced to a minimal diagram by a sequence of connected diagrams related by

- $2 \leftrightarrow 2$ moves;
- $1 \to 0$ moves;
- Dropping move 1  $\to$; and
- Dropping move 2  $\to$.

Since this version stays entirely within connected diagrams, it can also be stated in the dual language of hexagons. We will not prove this version here. It can be deduced as a corollary of Theorem 6 by analyzing the diagram just before applying a $1 \to 0$ move which disconnects the diagram.

There is also an internal characterization of which triple point diagrams are minimal.

Theorem 7. *A connected triple point diagram is minimal if and only if it has no strands which intersect themselves (*monogons*) or pairs of strands which intersect at two points, x and y, with both strands oriented from x to y (*parallel bigons*).*

The proof is in section 4. Note that a disconnected component is either a simple loop or has a monogon so the restriction to connected diagrams is not a large restriction.

This is an analogue of the theorem that a generic immersion of arcs in the plane has a minimal number of crossing points for the given pairing of the boundary (and corresponds to a tiling by rhombuses) iff there are no monogons or bigons. Note that in the triple point diagrams in Figure 2 you can see many bigons, but all of these bigons are anti-parallel: if the intersections are at A and B, one strand runs from A to B and the other strand runs from B to A.

History. The first version of this paper was completed and posted on the mathematics arXiv in 2004. The present version has a few minor updates, including an improved proof of Theorem 9 and motivation for Conjecture 21. After the first version of this paper was completed, I learned that Alexander Postnikov independently arrived at many of the same results in a somewhat different language. In particular, [Pos06, Theorem 12.7] is closely related to Theorem 5, with his move (M1) corresponding to our $2 \leftrightarrow 2$ move.

Acknowledgements. I would like to thank my advisor, Vaughan Jones, for suggesting this problem, and for his patience in waiting for me to write it up and then publish it. I am very pleased to be able to publish this paper in a volume celebrating his birthday.

I would also like to thank Vladimir Arnol'd, Sergei Fomin, Andre Henriques, Michael Polyak, Alexander Postnikov, Jim Propp, Chung-chieh Shan, and Vladimir Turaev for many helpful discussions and comments.

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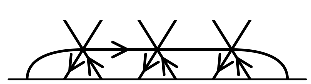
2. Generating matchings

In order to prove Theorem 3, we will explicitly construct a triple diagram that achieves a given pairing of in- and out-endpoints. The triple diagrams we construct turn out to be minimal diagrams. We will use this construction in the sections that follow: we will show that any two diagrams are related by reducing both of them to a diagram of the form we construct here.

We are given a circle with $2n$ endpoints around the boundary, alternately marked "in" and "out", and a pairing of the in-endpoints with the out-endpoints. Each pair of an in-endpoint and an out-endpoint divides the circle into two *intervals*. These intervals are ordered by inclusion. Pick a minimal interval I with respect to this order. Now start to construct the triple diagram by running a *boundary-parallel* strand S along I, introducing a triple crossing for each pair of strands that we cross over:



Note that there will always be an even number (possibly zero) of strands to cross over because of the alternation of in- and out-endpoints.

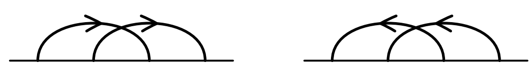
Remove S from our original pairing and swap the pairs we crossed over. The in- and out-endpoints in the new pairing still alternate and there are fewer strands. Continue by induction until all the endpoints are paired up.

A diagram constructed in this way, for some sequence of choices of minimal intervals, is called *standard*. We will see later (Corollary 13) that standard diagrams are minimal.

Remark. We could be a little more liberal in which intervals we allow, while still preserving minimality of the resulting triple crossing diagram. We can distinguish between the two intervals arising from a given matching according to whether the interval runs *clockwise* or *counterclockwise* from the in-endpoint to the out-endpoint. If, in the construction above, we always pick a clockwise interval that is minimal among all clockwise intervals, or a counterclockwise interval minimal among all counterclockwise intervals, the resulting triple diagram is still minimal.

We can also count the number of crossings in a standard diagram using only the connectivity information of the strands. For this purpose, pick any linear ordering on the strand endpoints that is compatible with the cyclic ordering. (That is, break the circle open at a basepoint, and lay it out on a line.)

Definition 8. A strand in a triple diagram from a to b (with a choice of basepoint) is *right-moving* if $a < b$, and is *left-moving* if $b < a$. Two strands S in a triple diagram are *linked* if their endpoints interleave around the circle. If the strands are linked, they are *parallel* if they are both left-moving or both right-moving, and are *anti-parallel* if one is left-moving and one is right-moving. Explicitly, two strands are parallel linked if their endpoints are in one of these two patterns:



(Note that these definitions refer to the strands, but they depend only on the connectivity.)

Theorem 9. *The number of triple crossings in a standard triple diagram D is equal to the number parallel linked of pairs of strands.*

Lemma 10. *For a given strand connectivity, the number of parallel linked pairs of strands doesn't depend on where you cut the circle open (i.e., which linear ordering compatible with the cyclic ordering you pick).*

Proof. Consider a strand S from a to b, with b the largest value in the linear ordering. S is necessarily right-moving. A strand that links S is right-moving if it has its out-endpoint between a and b, and is left-moving if it has its in-endpoint between a and b. Thus S is linked with an equal number of left-moving and right-moving strands, and we get the same number of parallel linked pairs of strands if we change the linear ordering so that b is the smallest value. Other cases are symmetric. □

Proof. We count the number of crossings in a standard diagram by induction on the number of strands. Pick a minimal interval, say from a to b. By Lemma 10, we may suppose the interval does not contain the basepoint. By symmetry, we may suppose $a < b$. Now consider what happens when we run a boundary-parallel strand S from a to b crossing a pair of adjacent out-endpoint c and in-endpoint d. Let the two other strands be T and U, so that S is linked with a strand T from another point e to c and a strand U from d to another point f. Let the diagram (with one fewer strand) above S be D'. Consider the location of e

and f relative to a and b. (Note neither e nor f can be between a and b, since T and U are linked with S by the assumption that $[a, b]$ was a minimal interval.)

Case 1. $e < a$**,** $f < a$**:** The linking between S and T is parallel, the linking between S and U is anti-parallel, and any linking created or destroyed between T and U is anti-parallel.

Case 2. $b < e$**,** $b < f$**:** The linking between S and T is anti-parallel, the linking between S and U is parallel, and any linking created or destroyed between T and U is anti-parallel.

Case 3. $f < a < b < e$**:** The linkings between S and T and S and U are both anti-parallel, while T and U are parallel linked in D but not in D'.

Case 4. $e < a < b < f$**:** The linkings between S and T and S and U are both parallel, while T and U are parallel linked in D' but not in D.

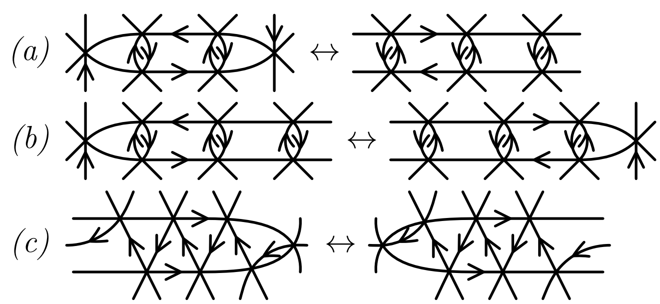
In each case, the number of parallel linked strands in D' decreased by 1 relative to D as a result of this crossing. □

3. Reducing diagrams

In order to prove Theorems 5 and 6, we will start with an arbitrary diagram and reduce it to one of the form described in Section 2 by straightening the strands one-by-one along the boundary.

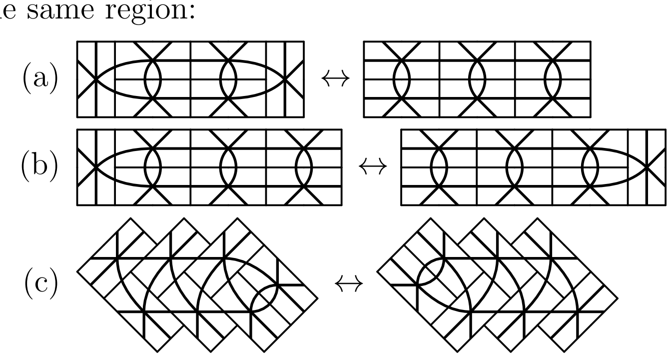
The basic blocks in the proof are some relations which let us slide one strand past another.

Lemma 11. *Each of the following triple-point diagrams can be related by sequences of* $2 \leftrightarrow 2$ *moves:*

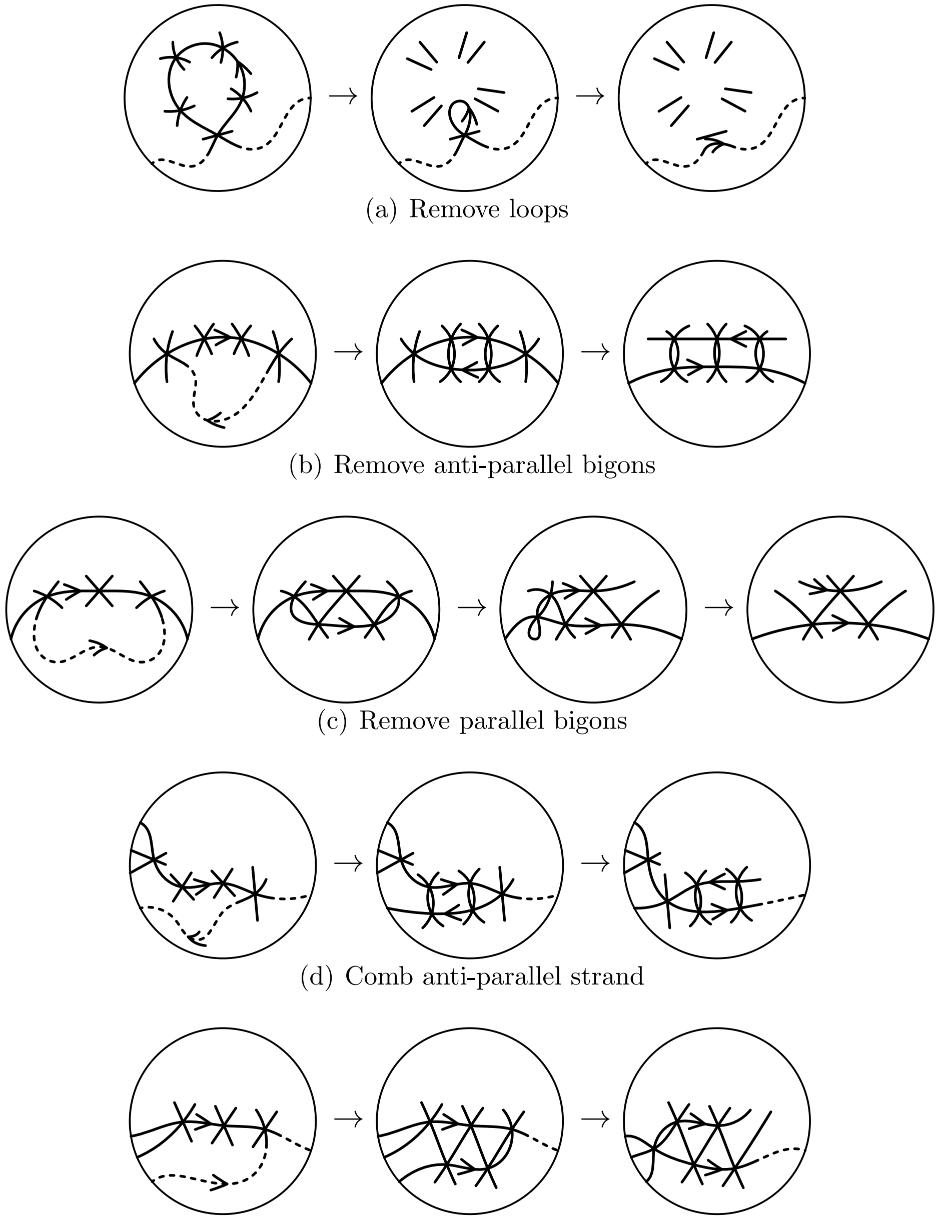


In each diagram the central section can be repeated an arbitrary number of times.

Proof. In each case, the two triple-point diagrams come from two different domino tilings of the same region:



We use the fact that any two domino tilings of the same region can be related by a sequence of domino flips. (It is easy to find an explicit sequence of flips in these cases.) □



(a) Remove loops

(b) Remove anti-parallel bigons

(c) Remove parallel bigons

(d) Comb anti-parallel strand

(e) Comb parallel strand

FIGURE 3. Steps in the proof of Lemma 12

Lemma 12. *Let D be a triple diagram and let I be a minimal interval of the associated pairing. Then D is related, by a sequence of $2 \leftrightarrow 2$, $1 \to 0$ and loop-dropping moves, to a diagram D' in which there is a strand boundary-parallel along I.*

Proof. We proceed by induction on the number of crossings and number of loop components in D, repeatedly applying the lemma to regions contained inside of D. We successively straighten the strand S which connects the endpoints of I.

Step 1. Remove self-intersections of S: If S has self-intersections, consider an innermost loop L of S: a loop of S which does not intersect itself, although it may intersect other portions of S. In Figure 3(a), L is shown solid and the rest of S is dashed. Consider the region contained inside L, including all triple crossings on the boundary of L except for the self-intersection crossing itself. This region has fewer triple crossings than the original diagram D (since we omitted the self-intersection), and the short interval between the two points where L intersects the boundary of the region is minimal, so by induction we can apply a sequence of $2 \leftrightarrow 2$, $1 \to 0$ and loop-dropping moves to reduce L so that it is boundary-parallel inside the region. A single further $1 \to 0$ move removes the original self-intersection of S.

Repeat this step until no self-intersections of S remain.

Step 2. Remove double intersections with S: Consider the region R enclosed by the strand S and the interval I, not including the triple crossings along S itself. If there is some strand that intersects S at least twice, then consider a strand segment T that gives a minimal interval along the boundary of R. Because there are fewer triple crossings inside R than there were in D, we can by induction apply moves until T is boundary-parallel to R. If T and S form an anti-parallel bigon, we can apply Lemma 11(a) as in Figure 3(b) to remove the bigon. If T and S form a parallel bigon, we can similarly apply Lemma 11(c) and two $0 \to 2$ moves as in Figure 3(c) to remove the bigon.

Repeat this step until there are no strands that intersect S twice. The number of triple intersections along S strictly decreases at each step, so this terminates.

Step 3. Comb out triple crossings: At this point, the only strands left between S and I come from the boundary within the interval I and cross S. (Here we use the minimality of I: no strands connect I to itself.) We wish to make S boundary parallel: that is, we want all the strands starting from I run directly to S, without any triple crossings first. Consider the strand T with the leftmost endpoint along I which violates this condition. By the hypothesis on T and the minimality of S, the interval corresponding to T is minimal inside the region R between S and the boundary, omitting the strands to the left of T and the triple crossings along S. Apply induction inside R to reduce T until it is boundary parallel. If T is oriented anti-parallel to S, apply Lemma 11(b) to make T run straight to S, as in Figure 3(d); if T is oriented parallel to S, instead apply Lemma 11(c), as in Figure 3(e).

Repeat this step until every strand runs directly from I to S.

Step 4. Remove disconnected components: Now S is boundary parallel, with the exception of possible disconnected components of the diagram between S and the boundary. For each disconnected component, consider a region R that encloses all of the component except for one arc T on the boundary of the component. The diagram contained inside R has no more triple crossings than D and at least one fewer loop, so by induction we can reduce it until the unique arc runs boundary parallel on the side facing T. This strand forms a simple loop with T, which we can drop.

Repeat this step until there are no disconnected components between S and the boundary, and so S is boundary parallel. The total number of loops in the diagram decreases at each step, so this terminates. $\square$

Proof of Theorem 5. Start with two minimal diagram D and D'. By repeatedly applying Lemma 12 to D, we can obtain a standard diagram D'' as in Section 2. Since D is minimal, we did not use any $1 \to 0$ moves in this process. Furthermore, any loop dropping move is necessarily preceded by a $1 \to 0$ move since D is connected and $2 \leftrightarrow 2$ moves preserve connectivity. Therefore we only used $2 \leftrightarrow 2$ moves. In the same way D' can be connected to D'' by a sequence of $2 \leftrightarrow 2$ moves, and so D and D' can be connected by a sequence of $2 \leftrightarrow 2$ moves, as desired. $\square$

Corollary 13. *Any standard diagram is minimal.*

Proof. This was proved during the proof of Theorem 5. $\square$

Proof of Theorem 6. By repeatedly applying Lemma 12, an arbitrary diagram can be reduced to a standard diagram by a sequence of $2 \leftrightarrow 2$, $1 \to 0$, or loop dropping moves. $\square$

Remark. The proof of the connectivity of tilings of a region by dominoes [Thu90] exploits a *height function* defined on the vertices of the tiling which changes by ± 1 along each edge of the tiling and by ± 3 along edges of the square grid which are not edges of the tiling. To show the space of domino tilings is connected, transform a diagram to an extremal one with minimal height function by repeatedly looking for a vertex which is a local maximum of the height function and performing a domino flip at that vertex. The extremal diagrams with respect to the height function are also standard diagrams in our terminology. Furthermore, general triple diagrams admit a multi-dimensional height function on the regions of the diagram which changes by 1 in one coordinate when you cross an edge. (See [HS10, Section 1.2].) Standard diagrams are extremal with respect to an appropriate projection from the multi-dimensional height function to $\mathbb{R}$. Both proofs involve connecting an arbitrary tiling to an extremal one.

There is a corresponding multi-dimensional cube recurrence related to rhombus tilings, studied by Henriques and Speyer [HS10].

4. Minimal diagrams

Minimal triple diagrams play a somewhat special role in the proof above. In order to get a better understanding of minimal diagrams, we will prove Theorem 7

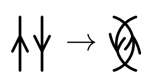
Definition 14. A *badgon* in a general diagram of oriented curves, not necessarily a triple diagram, is a monogon, parallel bigon, or simple loop.

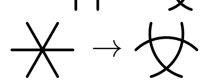
Theorem 7 says that a diagram is minimal if and only if it has no badgons.

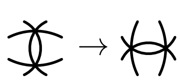
Proof of Theorem 7. First suppose a diagram D has a monogon or parallel bigon. In either case, we can straighten the strand as in the proof of Lemma 12, Steps 1 and 2 (Figures 3(a) and 3(c)) respectively. At the end of the straightening (and possibly earlier), there are one or two $1 \to 0$ moves, which strictly reduces the number of triple crossings; thus D was not minimal.

For the other direction, we need a lemma.

Lemma 15. *Let D and D' be general diagrams, not necessarily connected or with only triple crossings, related by one of the following moves:*

(a) *An anti-parallel self-tangency involving distinct strands* *;*

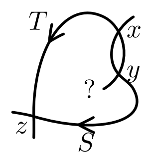
(b) *Perturbing a triple crossing into 3 double crossings* *; or*

(c) *A $2 \leftrightarrow 2$ move* *.*

Then D and D' either both have no badgons or both have at least one badgon.

Proof. We consider each case in turn.

(a) There are two crossings created by the self tangency; call them x and y. All other crossings are the same in the two diagrams. Any badgon that does not involve x or y will be the same on the two sides of the diagram, so let us suppose there is none. Because the two strands are different, there is no monogon created in D'. Because the self-tangency is anti-parallel, we can only have a parallel bigon in D' involving both x and y if one of the two strands is a loop, which necessarily has badgon in D as well. The only remaining possibility for a badgon in D' that is not in D is a parallel bigon involving (wlog) x and another crossing elsewhere in the diagram, say z, like this:



In D', one strand S runs from x through y to z; the other strand T runs in one direction from x to z. In the other direction T runs from x to y. After the intersection with y, T is inside the bigon formed by S and T and must exit somewhere. If it exits by crossing T, there is a monogon; if it exits by crossing S, there is another parallel bigon. In either case, this new badgon exists in both D and D'.

(b) If any two of the strands involved in the triple crossing are the same, there is a monogon in both D and D'. Otherwise, a badgon in D' will only involve at most one of the three crossings and there gives a badgon in D as well, and vice versa.

(c) If any two of the strands involved are the same, there is a badgon on both sides of the diagram, so let us suppose all strands are distinct. The $2 \leftrightarrow 2$ move can be decomposed into a sequence of the moves above: perturb the two triple crossings, perform two anti-parallel self-crossings, and collect the crossings into two new triple crossings. None of these steps change the presence of badgons. □

We now finish the proof of Theorem 7. By Theorem 6, any connected non-minimal diagram D can be turned into a minimal diagram by a sequence of moves, necessarily involving a $1 \to 0$ move. Let D' be the diagram just before the first $1 \to 0$ move. D' has a monogon at the site of the $1 \to 0$ move. D and D' are related by a sequence of $2 \leftrightarrow 2$ moves, so by Lemma 15, D has a badgon as well. □

5. Connections and future directions

5.1. **Invariants of plane curves and Legendrian knots.** One natural connection is with Arnol'd's theory of invariants of plane curves [Arn94]. Deforming a generic smooth oriented plane curve in a generic way, you pass through three types of non-generic points (what Arnol'd calls *perestroikas*): triple crossing, direct self-tangency, and inverse self-tangency. If an invariant of generic plane curves does not change under a triple crossing, it can be extended naturally to an invariant of curves with triple points. If, in addition, it does not change under an inverse self-tangency, it will not change under the domino flip move.

For instance, Arnol'd's J_+ invariant of plane curves, the simplest invariant unchanged under these two operations, when evaluated on a curve with only alternating triple points and no double points gives the number of triple points plus the index of the curve, plus or minus 1 (depending on the orientation on the boundary of the exterior region of the plane).

One way to construct invariants of plane curves that do not change under a triple crossing or an inverse self-tangency is by taking an invariant of the *Legendrian knot* in $\mathbb{R}^2 \times S^1$

obtained by lifting the curve to the unit tangent bundle of the plane. This connection with Legendrian knots leads to some natural questions:

Question 16. Which Legendrian knots have planar representations with only alternating triple points?

Remark. The answer to this question is not "all", since the J_+ invariant must be strictly positive for non-trivial knots represented by an alternating triple diagram.

Question 17. Can two different triple diagrams for a single Legendrian knot always be related by $2 \leftrightarrow 2$ moves?

5.2. **Cluster algebras and the octahedron recurrence.** (This section is joint work with Andre Henriques.)

There are two natural cluster algebras as introduced by Fomin and Zelevinsky [FZ02] associated to triple diagrams with a given connectivity. Loosely speaking, a *cluster algebra* A of rank n is a commutative algebra with unit and no zero divisors with a distinguished set of *clusters*: sets of elements $x_1, \dots, x_n \in A$ so that every element in A can be written as a Laurent polynomial in the x_i in any given cluster. (A Laurent polynomial in the x_i is a polynomial in the x_i and x_i^{-1}.) Furthermore, any two clusters can be connected by a sequence of *exchange relations*: replacing one element x_i of a cluster by a conjugate variable $\tilde{x}_i$, related by

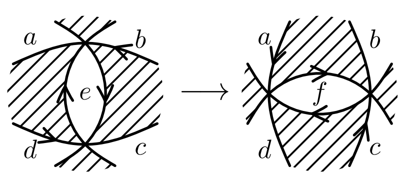
$$x_i \tilde{x}_i = M_1 + M_2$$

where M_1 and M_2 are monomials in the $n-1$ other elements of the cluster. (This is a description of cluster algebras, not a definition.)

Each connectivity class of triple diagrams gives a cluster algebra, with each triple diagram giving a cluster. The variables in the cluster correspond to the white regions in the checkerboard coloring of the triple crossing diagram.

When you perform a $2 \leftrightarrow 2$ move, the cluster variables are unchanged if the region in the center of the move is black. If the region is white, the conjugate variable to the variable in the bigon is given by

(1)



where

$$f = \frac{ac + bd}{e}.$$

One initially surprising fact is the *Laurent phenomenon*: start from an arbitrary triple diagram and apply a number of these $2 \leftrightarrow 2$ moves. Express the variables on each region in terms of the variables on the original regions using the exchange relation. Each variable will be a Laurent polynomial! The number of terms in these Laurent polynomials becomes quite large, and you have to divide by earlier polynomials in the exchange relation, but in each case the division comes out exactly. Furthermore, the coefficients of the Laurent polynomials are positive integers.

This cluster algebra turns out to be a disguised version of the n-dimensional octahedron recurrence [HS10, Section 1.2]. The coefficients are conjecturally positive, but there is not currently a combinatorial interpretation in all cases. For the 3-dimensional octahedron recurrence, a combinatorial interpretation in terms of matchings was given by David Speyer [Spe07].

Postnikov [Pos06, Sco03] has used closely related diagrams to parameterize bases for totally positive cells in Grassmannians. Each connectivity corresponds to a particular totally positive cell; each triple crossing diagram gives a basis for the Grassmannian. The bases corresponding to triple crossing diagrams that are related by a $2 \leftrightarrow 2$ move are again related by (1). See [Pos06, Section 14] for a description of the relation.

5.3. **Relations between relations.** If Theorem 3 guarantees that triple diagrams generate all possible permutations and Theorem 6 guarantees that the $2 \leftrightarrow 2$, $1 \to 0$, and loop dropping moves give all relations between triple diagrams, the next natural question is what the relations between relations are. For simplicity, we will restrict ourselves to minimal triple diagrams.

Before stating a conjecture, we first pick out some particularly strand connectivities to consider.

Definition 18. Fix an integer $n \geq 1$ and an odd integer $1 \leq k \leq 2n-1$. The (n,k) *strand connectivity* is the connection of strands that connects each input to the output position that is k steps counterclockwise around the outside of the diagram. (Since k is odd, this connects each input strand to an output strand.) Here are some examples of connectivities and their minimal diagrams.

- In the connectivities $(n,1)$ and $(n,2n-1)$, each strand is connected to a neighbor, and we get a diagram with no triple-crossings.
- The connectivity $(3,3)$ gives a single triple-crossing.
- In the connectivities $(4,3)$ and $(4,5)$, there are two minimal diagrams, each with two triple-crossings. These two diagrams are the two sides of the $2 \leftrightarrow 2$ move.
- The minimal diagrams for the connectivities $(5,3)$ and $(5,5)$ are show in Figure 4.

Proposition 19. *The number of triple-crossings in a minimal diagram of connectivity* $(n, 2\ell+1)$ *is* $\ell(n-\ell-1)$.

Proposition 20. *Among all connectivities for n-strand triple-crossing diagrams, those with the maximal number of triple-crossings are*

- *if n is odd: the connectivity (n,n), with $(n-1)^2/4$ triple-crossings, and*
- *if n is even: the connectivities $(n,n-1)$ and $(n,n+1)$, with $n(n-2)/4$ triple-crossings.*

Proof of Propositions 19 and 20. Straightforward from Theorem 9. □

Conjecture 21. *For triple diagrams with a given connectivity, consider the 2-complex C with vertices given by minimal triple diagrams, edges given by $2 \leftrightarrow 2$ moves, and 2-cells of the following types:*

(1) Squares for two different $2 \leftrightarrow 2$ moves in non-interfering parts of the diagram;
(2) Pentagons as on the left of Figure 4 or their orientation reverse, from connectivities $(5,3)$ and $(5,7)$; and
(3) Decagons as on the right of Figure 4, from connectivity $(5,5)$.

Then C is simply connected.

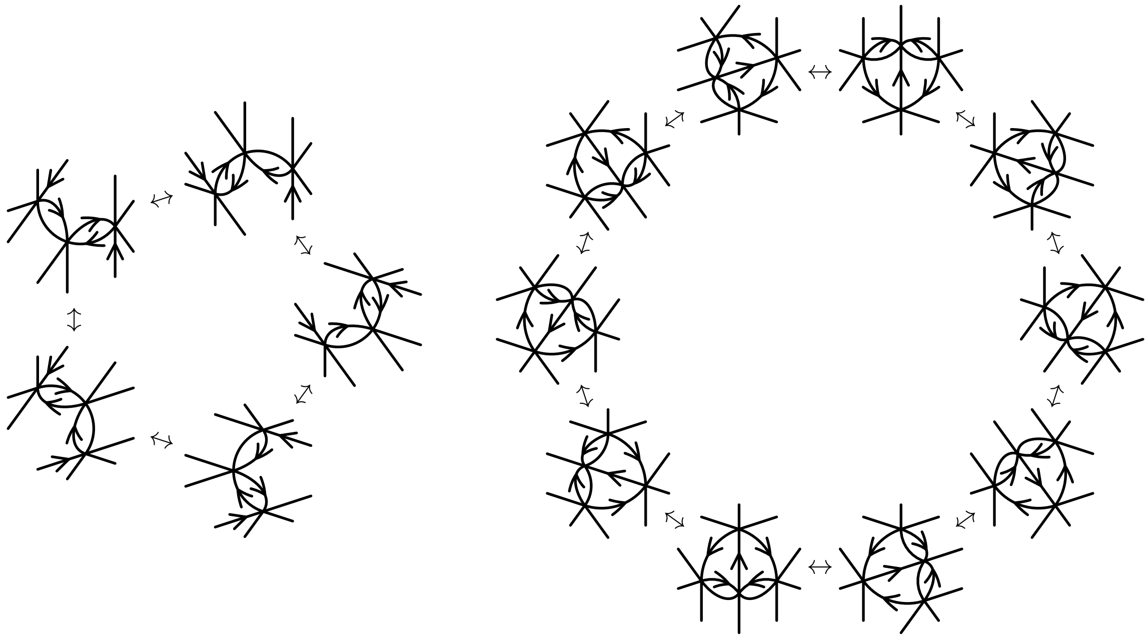


FIGURE 4. Conjectured relations between relations

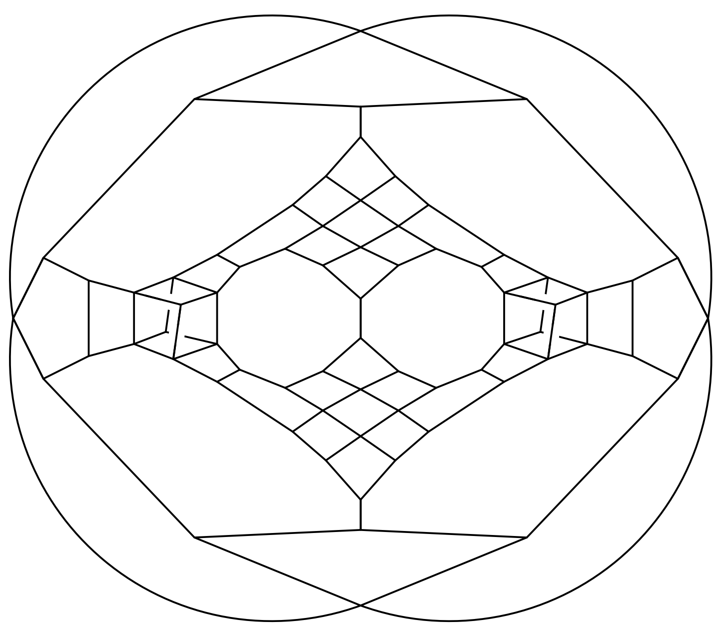
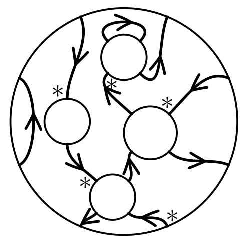


FIGURE 5. The graph of minimal triple-diagrams for connectivity $(6, 5)$. Note the presence of two cubes. The graph has 3-fold rotational symmetry around these cubes.

For instance, the complex $C(n, 3)$ for connectivity $(n, 3)$ is the 2-skeleton of the associahedron K_n, which is simply-connected. The complex $C(6, 5)$ is shown in Figure 5, and is again simply-connected.

More generally, one could conjecture that there is a contractible complex with fundamental n-cells given by the diagrams of connectivity $(n+3, k)$ for k an odd number between 3 and $2n+3$. As in Conjecture 21, there are also products of lower-dimensional cells.

5.4. **Planar algebras.** The original motivation for considering triple crossings came from *planar algebras* as introduced by Jones [Jon99]. Briefly, planar algebras in the form we consider have a complex vector space B_n for each n, the space of *n-boxes*. Each n-box conceptually has $2n$ strands attached to the boundary, alternating in and out. They may be joined together by oriented *arc diagrams* like this one:



This diagram, for instance, has 4 inputs (the little circles), with 4, 2, 4, and 4 strands, respectively; the output (the large circle) has 8 strands. This diagram therefore gives a multilinear map

$$B_2 \otimes B_1 \otimes B_2 \otimes B_2 \to B_4.$$

Each arc diagram gives a similar multilinear map, and if you plug the output of one multilinear map into the input of another, you get the multilinear map corresponding to gluing together the two arc diagrams. Formally, this is the structure of a (typed) operad. (To be precise, you must also provide some markings to know the orientations when you plug in elements.)

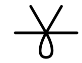
Typically one assumes that B_0 is 1-dimensional and is identified with $\mathbb{C}$. In particular, this implies that a simple loop, which can be thought of as an element of B_0, has some definite value $\delta \in \mathbb{C}$.[1]

One natural way to start to classify planar algebras is by their generating sets. Every planar algebra contains the *Temperley-Lieb algebra*, consisting of diagrams with only non-crossing arcs and no inputs. Non-crossing diagrams with no loops give us $1, 1, 2, 5, 14, \dots$ elements inside B_n for $n = 0, 1, 2, 3, 4, \dots$. Most planar algebras are not spanned by the Temperley-Lieb algebra, but are generated by the addition of some extra elements. For example, Bisch and Jones [BJ00, BJ03] have studied planar algebras that are generated by a single 2-box. Consider instead planar algebras where the 0-, 1-, and 2-boxes are spanned by Temperley-Lieb elements and which are generated by a single 3-box. If we draw the 3-box as a triple crossing,[2] then every triple diagram gives an n-box. If we assume further that the dimension of B_4 is less than or equal to 25 and that the relations are "generic" and "symmetric" in senses to be made precise below, the results in this paper imply that the dimension of B_n is less than or equal to $n!$ for all n.

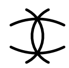
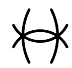
To prove this, consider diagrams modulo "simpler" diagrams that either have fewer loops or fewer triple crossings, and look at the three moves used in Theorem 6.

[1] One of the subtleties we're ignoring is that a clockwise and counterclockwise loop could potentially have different values.

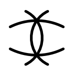
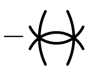
[2] Another subtlety we're ignoring here is that the 3-box need not be symmetric under rotation by 120 degrees, as the triple crossing is. This turns out not to affect the subsequent discussion.

- A loop is equal to a constant δ.
- Because B_2 is spanned by Temperley-Lieb elements, the diagram  is equal to a linear combination of diagrams with no crossings, giving the $1 \to 0$ move modulo lower order terms.
- There are 26 elements of B_4 obtained from the 14 elements of the Temperley-Lieb algebra, the 8 ways of adding a single arc to a single triple crossing, and the 4 ways of connecting two triple crossings along two adjacent arcs, so by assumption there is at least a 2-dimensional space of relations between these 26 elements. The highest terms are of the last kind. Assume that the relations are generic, in the sense that when we take the quotient by the space spanned by the first 22 diagrams, the space of relations is a nontrivial subspace W of the 4-dimensional space V with basis given by the remaining 4 diagrams.

  Rotating by 90 degrees gives a symmetry of order two of V that must preserve W. V decomposes into two subspaces $V = V^+ \oplus V^-$, both of dimension two. If we furthermore assume that W is symmetric when we reverse all the arrows, W must contain either V^+ or V^-. Then we have either

(2)  $=$  $+$ (lower order terms)

or

(3)  $= -$  $+$ (lower order terms),

respectively.

The loop in Figure 4(a) shows that Equation (3) implies that the entire space of diagrams of two triple crossings vanishes modulo lower order terms. Thus the three moves involved in Theorem 6 are all true modulo lower order terms. Therefore, modulo lower order terms,

- Non-minimal diagrams are equal to 0, and
- Minimal diagrams with the same connectivity are equal to each other.

(We need the fact that you don't need to increase the size of the diagram in Theorem 6.) Therefore B_n is spanned by $n!$ minimal diagrams, one for each connectivity.

One source of planar algebras satisfying these hypotheses is the skein module of the HOMFLYPT polynomial, a well-known invariant of oriented links. You might expect a knot invariant to give a planar algebra based on a simple crossing, which would be a 2-box; however, the orientations in the HOMFLYPT polynomial and the requirement that the strands alternate in-out force you to consider triple crossings.

Since the first version of this paper was published, this approach has been worked out by Jones, Liu, and Ren [JLR16].

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