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Truncation of Wavelet Matrices: Edge Effects and the Reduction of Topological Control

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ABSTRACT

Edge effects and Gibbs phenomena are a ubiquitous problem in signal processing. We show how this problem can arise from a mismatch between the "topology" of the data D (e.g., an interval in the case of a time series or a rectangle in the case of a photographic image) and the topology X (often a circle or torus) natural to the construction of the transformation O. The notion of a manifold control space X for an orthogonal transformation O is introduced. It is proved that no matter how complicated X is, O may be "truncated" to an O' with control space D, homeomorphic to an interval or a product of intervals. This yields a new, topologically motivated approach to edge effects. We give the complete details for applying this approach to the discrete Daubechies transform of functions on the unit interval so that no data are wrapped around from one end of the interval to the other.

The discrete operation $v \to Ov$, multiplication of a data vector v (or partially processed vector) by an orthogonal transformation O, lies at the

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heart of linear signal processing. It is fortunate that O will often be quite sparse. In wavelet theory the algebraic conditions which define O may create a topologically complex pattern of sparsity. For example, in the 1-dimensional discrete wavelet transform the pattern is circular; in dimension n the pattern is modeled on an n-torus R^n/Z^n (the n-dimensional real vector space modulo the lattice of integer points).

In processing a time series (domain an interval) or a photographic image (domain a rectangle), and in many other applications, there is no useful relation between various data near different parts of the boundary. In such cases O can be modified, or *truncated*, to a new orthogonal matrix O' agreeing in most of its entries with O but such that the passage $v \to O'v$ does not involved the formation of linear combinations of data gathered at far away points near the domain boundary. This represents an approach to edge effects somewhat different from the usual windowing methods.

There is a fairly general setting—linear algebra with metrical control (see for example [7])—for the problem of passing from O to O'. But before turning to this, we give a detailed description of truncation as it applies to the 1-dimensional Daubechies wavelets [2]. For the simplest nontrivial example, the truncation is given explicitly. Finally we return to the general case but allow the 2-torus T^2 to stand in for an arbitrary metric space. This substitution relieves the reader of a general discussion of triangulations and their dual handle structures while presenting enough of the general picture that other (and higher dimensional) cases should amount to a manageable exercise. We treat only the dead zero version of sparsity, but expect analogous results to hold with only assumptions of rapid decay.

Consider the discrete (1-dimensional) Daubechies transforms:

$$M_{2n}^{2^N} = M_{2n}$$
$$= \begin{bmatrix} c_0 & \cdots & \cdots & c_{2n-1} & 0 & 0 & \cdots & \cdots & 0 \\ c_{2n-1} & -c_{2n-2} & \cdots & -c_0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & c_0 & \cdots & & c_{2n-1} & \cdots & \cdots & 0 \\ \vdots & & c_{2n-2} & -c_{2n-2} & \cdots & -c_0 & \cdots & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ c_2 & \cdots & c_{2n-1} & & & & & c_0 & c_1 \\ c_1 & \cdots & c_0 & & & & & c_{2n-1} & -c_{2n-1} \end{bmatrix}.$$

M_{2n} denotes a $2^N \times 2^N$ matrix for $2^N \gg 2n$. Orthogonality of M_{2n} is implied by n equations of the form

$$\sum_i c_i c_i = 1,$$
$$\sum_i c_i c_{i+2k} = 0, \qquad 1 \leqslant k \leqslant n-1. \tag{1}$$

Additionally, n-moment conditions may be enforced:

$$\sum_i (-1)^{i+1} i^k c_i = 0, \qquad 0 \leqslant k \leqslant n-1. \tag{2}$$

Clearly the nonzero entries of M_{2n} are organized by the geometry of the circle. This is a consequence interpreting the subscripts in (1) modulo $2n$. This interpretation derives from translation invariance and corresponds to taking wraparound boundary conditions on the data vector v.

There is a process known as a pyramid [2; 4; 6, Chapter 13.10] into which v may be fed. First, form $M_{2n}^{2^N}(v)$, and set aside even entries (numerically these are the finest scale wavelet coefficients of v). Second, form $M_{2n}^{2^{N-1}}$ [odd entries of $M_{2n}^{2^N}(v)$], and set aside the resulting even entries (these are the coefficients of the next to finest scale). Continue forming $M^{2^{N-k}}$ [odd entires of $M_{2n}^{2^{N-(k-1)}}(v)$]. The even entries for $k = 0, 1, \ldots, 2^{N-k} \geqslant 4n$ give numerical approximations to the wavelet coefficients of v. This pyramid computes the Daubechies transform in the discrete setting and defines the continuous Daubechies transform as a limit where $N \to \infty$ and $N - k =$ constant.

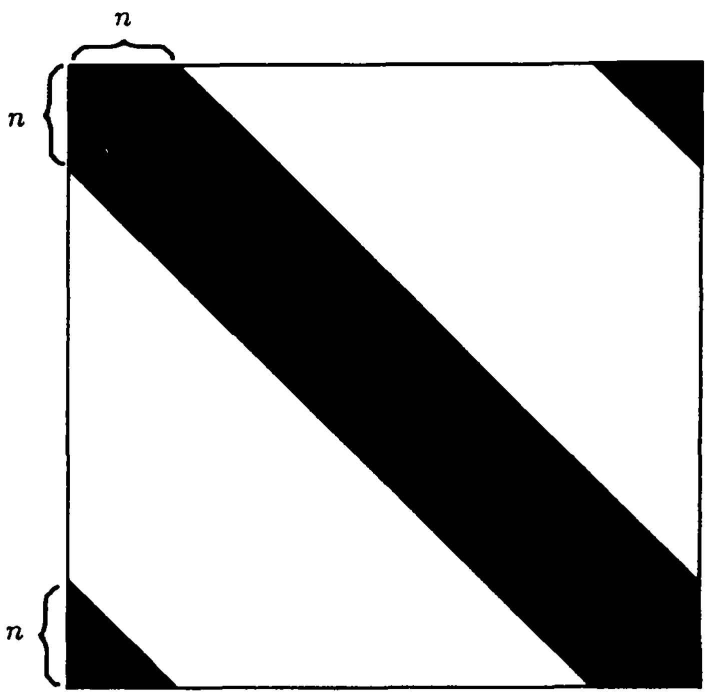
We give a precise procedure for constructing an orthonormal truncation M'_{2n} when $4n < 2^N$. In the more trivial case $4n \geqslant 2^N$, M_{2n} is not modified: $M'_{2n} = M_{2n}$.

M_{2n} will be truncated to a purely band-diagonal $2^N \times 2^N$ matrix M'_{2n} which agrees with M_{2n} except in the first n and last n rows. The procedure requires the orthogonality relations (1) to hold for M_{2n} but makes no use of the moment conditions (2), and it is quite possible in applications that the free parameters used up in the condition (2) should instead be saved for some different optimization more suited to the bounded setting.

Using the primed matrices M'_{2n} in the pyramid, we obtain as output a discrete version the Daubechies transform adapted to $L^2[0,1]$. Our adaptation is quite different (and less developed, since we have not fixed $c_0, \ldots, c_{2n-1}$ completely) from the proposal in [1] and [5], but the two may be compared if we imagine our discrete procedure taken to the limit. They share the property that wavelet basis functions from $L^2(R)$ whose support is contained wholly in $[0,1]$ are unchanged. In our basis, wavelets whose supports [in $L^2(R)$] overlap a boundary point are increasingly crumpled and seemingly less smooth as the overlap increases. In all discrete stages each boundary value is carried by a simple "Dirac wavelet" (which disappears in the continuous limit), and all other wavelets vanish at the boundaries. In [1] the authors choose special functions adapted to and localized near the borders to complete their basis. The freedom in these choices enables [1] to produce wavelets on $[0,1]$ whose first $n-1$ moments vanish. While our boundary behavior may be less desirable in many applications, we continue to be interested in this basis because of the mathematically canonical nature of the truncation $M_{2n} \to M'_{2n}$. It is an open problem to optimize our wavelets by replacing the relations (2) with some unspecified relations.

For ease of description we henceforth assume n odd. The case of n even is nearly identical. Before we start, let us revise slightly the form of the basic wavelet matrix M_{2n}. Let us cycle the columns through $n - 1$ steps so that the nonzero entries are roughly centered on the diagonal and the pattern of nonzero entries is (roughly)

$M_{2n} =$



(3)

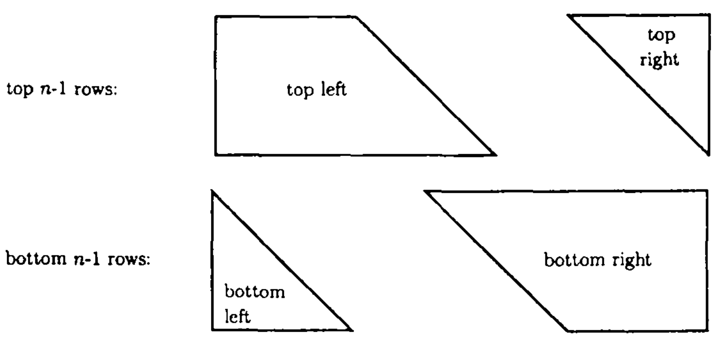
The new form of M_{2n} is preferable. When the M_{2n} of various sizes 2^N are inserted into the pyramid which computes wavelet coefficients, the *edge effects*—that is, the mixing of data obtained from opposite ends of the interval—will fall only at the last possible moment (rather than repeatedly) on data from the middle of the measurement stream.

The first unbroken row of M_{2n} [see (3)] is the nth when n is odd and the $n + 1$st when n is even (due to the staircase form near the diagonal). The "intact" rows of M_{2n},

$$\begin{aligned} r_n &= c_0, \ldots, c_{2n-1}, 0, 0, \ldots, \\ r_{n+1} &= c_{2n-1}, -c_{2n-2}, \ldots, -c_0, 0, 0, \ldots, \\ r_{n+2} &= 0, 0, c_0, \ldots, c_{2n-1}, \ldots, \\ r_{n+3} &= 0, 0, c_{2n-1}, -c_{2n-2}, \ldots, -c_0, \ldots, \\ &\vdots \\ r_{2N-n+1} &= 0, \ldots, c_{2n-1}, -c_{2n-2}, \ldots, -c_0, \end{aligned} \tag{4}$$

span the "central" subspace $C \subset V$, where V is the underlying vector space on which M_{2n} operates.

Here is the algorithm for writing down M'_{2n}. The first and last $n-1$ rows are broken into a left and a right piece by a sea of zero entries. Geometrically speaking (and ignoring the staircase effect), these nonzero entries break up into two trapozoids and two triangles:



(5)

Call the $4(n-1)$ vectors represented by the rows of these figures $\{v_i^{t,l}, v_i^{t,r}, v_i^{b,l}, v_i^{b,r},\ 1 \leqslant i \leqslant n-1\}$, where t = top, b = bottom, l = left, and r = right.

To construct the first $n-1$ rows of M'_{2n} follow these steps. Place bottom left on top of top left to form an $2(n-1) \times 2(n-1)$ triangle. Select every other row of this triangle to form a $(n-1) \times 2(n-1)$ triangle of rows.

These rows are

$$v_2^{b,l}, v_4^{b,l}, \dots, v_{n-1}^{b,l}, v_2^{t,l}, v_4^{t,l}, \dots, v_{n-1}^{t,l}. \tag{6}$$

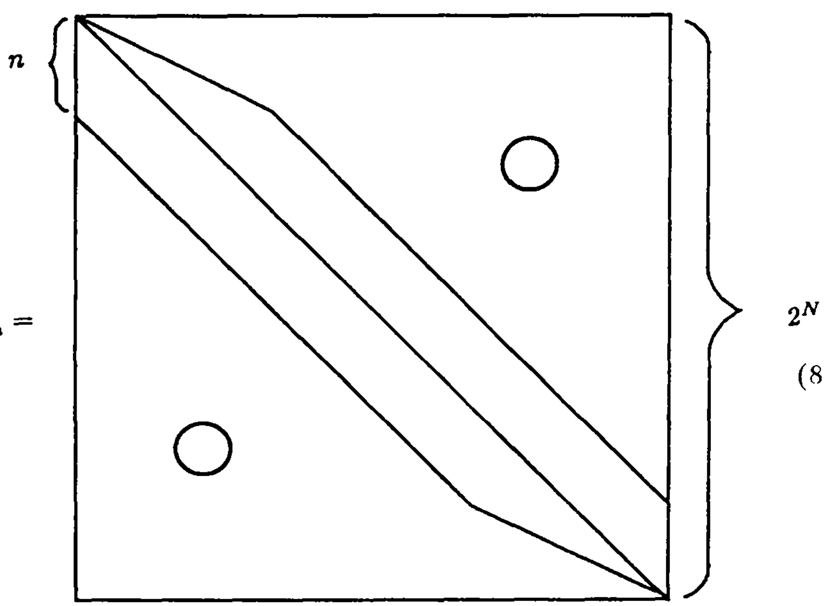
Starting with the shortest, apply the Gram-Schmidt orthonormalization to obtain another $(n-1) \times 2(n-2)$ triangle of rows $w_1^t, \dots, w_{n-1}^t$. Insert these as the first $n-1$ rows of M'_{2n}.

The middle rows of M'_{2n} agree with those of M_{2n}. The last $n-1$ rows of M'_{2n} are made by applying Gram-Schmidt (in reverse order, starting with the shortest rows) to

$$v_2^{b,r}, v_4^{b,r}, \dots, v_{n-1}^{b,r}, v_2^{t,r}, v_4^{t,r}, \dots, v_{n-1}^{t,r}. \tag{7}$$

The nonzero entries of M'_{2n} are indicated below (in particular, M'_{2n} is band-diagonal with no increase in band width over the central portion of M_{2n}):

$M'_{2n} =$



(8)

We must now explain why M'_{2n} is an orthogonal matrix. Let $C^{\perp}$ denote the subspace of vectors in V which are perpendicular to C. Since M_{2n} is orthogonal, the $2(n-1)$ rows $\{v_i^{t,l} + v_i^{t,r}, v_i^{b,l} + v_i^{b,r},\ i \leqslant i \leqslant n-1\}$ of M_{2n} span $C^{\perp}$. Thus

$$\left\{v_1^{b,l}, \dots, v_{n-1}^{b,l}, v_1^{t,l}, \dots, v_{n-1}^{t,l}\right\} \tag{9a}$$

and

$$\left\{v_1^{b,r}, \dots, v_{n-1}^{b,r}, v_1^{t,r}, \dots, v_{n-1}^{t,r}\right\} \tag{9b}$$

together will span $C^{\perp}$. Also the geometry of the matrix, when $n \ll 2^N$, implies that any vector in (9a) is perpendicular to any vector in (9b). Thus it suffices to verify that the span of the even numbered vectors in each set contains the odd numbered vectors in that set. In fact we will show using the orthogonality relations (1) that

(∗) The $2k+1$st and $2k+2$nd vectors in either set are dependent, modulo the sorter vectors, in that set.

Because of the symmetry we only consider the first set, which we write out below:

$$\begin{array}{llllllll}
\text{row } 1: & c_{2n-2} & c_{2n-1} & & & & & \\
\text{row } 2: & c_1 & -c_0 & & & & & \\
\text{row } 3: & c_{2n-4} & c_{2n-3} & c_{2n-2} & c_{2n-1} & & & \\
\text{row } 4: & c_3 & -c_2 & c_1 & -c_0 & & & \\
\text{row } 5: & c_{2n-6} & c_{2n-5} & c_{2n-4} & c_{2n-3} & c_{2n-2} & c_{2n-1} & \\
\text{row } 6: & c_5 & -c_4 & c_3 & -c_2 & c_1 & -c_0 & \\
\vdots & \vdots & & & & & & \\
\text{row } 2n-2: & c_{2n-3} & \cdots & \cdots & \cdots & \cdots & \cdots & -c_0
\end{array} \tag{10}$$

The two term relation in (1) implies the first two rows are collinear.

Clearly row 4 is perpendicular to $\alpha = (0, 0, c_0, c_1)$ and $\beta = (c_0, c_1, c_2, c_3)$. Row 3 is also perpendicular to α and β, using the two and four term relations (1). Since row 2 is also perpendicular to α and β, rows 2, 3, and 4 span only a 2-dimensional space. Since row 2 is collinear with neither row 3 nor row 4, it follows that rows 3 and 4 are dependent modulo row 2.

Similarly, rows 5 and 6 both lie in the 3-dimensional subspace perpendicular to $\alpha' = (0, 0, 0, 0, c_0, c_1)$, $\beta' = (0, 0, c_0, c_1, c_2, c_3)$, and $\gamma' = (c_0, c_1, c_2, c_3, c_4, c_5)$. Also in this subspace are rows 2 and 4. Since row 2 and row 4 are independent of each other and of rows 5 and 6, it follows that rows 5 and 6 are dependent modulo rows 2 and 4.

Continuing in this way, we see that row $2k + 1$ and row $2k + 2$ are dependent modulo the shorter even numbered rows, establishing (*).

A variant on our definition of M'_{2n} might be considered. Proceed as above except, when applying Gram-Schmidt, proceed from longest to the shortest rows. This doubles the band thickness of the matrix but has the advantage that the rows of M'_{2n} only change gradually as the upper and lower rows are approached. It would be interesting to see if this second definition yields more perspicuous wavelets.

Table 1 gives numerical results for the truncated matrices produced by this procedure, for the cases $N = 2, 3, 4, 5, 6$. (For typographical compactness, the table shows the transposed matrices, with orthogonal columns rather than orthogonal rows.) For $N = 2, 4, 6$ the table examplifies the slight difference of procedure required for the even N case: The first complete Daubechies coefficient vector is offset by one from the edge, resulting in a unit vector in the first row (or column).

Figures 4 and 5 show, for the case $n = 2$ (the simplest Daubechies wavelets) the result of the pyramid construction for both truncated and untruncated coefficient matrices. As an approximation to the continuum ($N = 10$), the pyramid operates on a vector of length 1024. One sees (Figure 1) that those wavelets whose support wraps around are significantly modified to adapted wavelets which do not wrap around, but nevertheless preserve orthonormality. Wavelets whose support does not wrap around (Figure 2) are unchanged in the continuum limit; for a finite hierarchy and the case of even N, there is a slight modification (here of order $\frac{1}{1024}$) due to the offset of the coefficient vectors by one. This allows the dashed and solid lines in Figure 2 to be both visible.It should be remarked that the truncation $M_{2n} \to M'_{2n}$ can be made directly applicable to image (or higher dimensional data) processing by using the "tensorial" or "product" wavelet decompositions on a rectangle (or multirectangle). For other more subtle wavelet bases in dimension 2 and higher, analogs of this truncation exist. It is not known if the details of these will prove as elegant—for example, some coarsening of the bands of nonzero entries may be expected corresponding to δ-control in the hypothesis and ϵ-control in the conclusion of the subsequent theorem. However, the existence of the truncation is assured by the theorem. The example we have just worked through deals with the passage from circular control to interval control. It is a first case of a rather flexible theory. We now turn to a more general setting in which truncation may be achieved.

Let X be a compact metric space (called the *control space*), and $\epsilon > 0$ a positive real number. Let V and W be finite dimensional inner product spaces with orthonormal bases $\{v_1, \ldots, v_a\}$ and $\{w_1, \ldots, w_b\}$. Let $f : \{v_1, \ldots, v_a\} \to X$ and $g : \{w_1, \ldots, w_b\} X$ be functions called *control functions*. We say that a linear map $h : V \to W$ is ϵ-small or ϵ-controlled (w.r.t the control functions) if the components of h (w.r.t. the given bases) satisfy $h_{ij} = 0$ whenever $\mathrm{dist}_X(f(v_i), g(w_j)) \geqslant \epsilon$.

Suppose the control space X is smooth k-dimensional Riemannian manifold[1] with the "path metric" obtained by integrating length along paths:

$$\mathrm{dist}_X(x, y) = \inf_{\substack{\text{paths in } X \\ \text{from } x \text{ to } y}} \int_\gamma \left\| \frac{d\gamma}{dt} \right\| dt. \tag{11}$$

Let $D^k \to X$ be a compatible piecewise linear map of a closed k-cell onto X which is an imbedding on its interior. D^k is called a top cell for X. Give D^k the path metric induced from paths in D^k, and assume all control maps have image $\subset \mathrm{int}\, D^k$. We have

$$\mathrm{dist}_{D^k}(x, y) = \inf_{\substack{\text{paths } \gamma \text{ in } D^k \\ \text{from } x \text{ to } y}} \int_\gamma \left\| \frac{d\gamma}{dt} \right\| dt. \tag{12}$$

Let X, with its path metric, be a δ-control space for a linear map $P : V \to V$. That is, $\{v_1, \ldots, v_a\}$ is a basis for V, and f a map $f\{v_1, \ldots, v_a\} \to X$ such that P is δ-small. A restriction of control to a subspace $Y \subset X$ is possible if $f\{v_1, \ldots, v_a\} \subset Y$. We write the map with restricted control $P\|_Y$ to distinguish it from the usual notation for restricting to a subspace $P|_W$, $W \subset V$. This simply means we will now measure distances according to paths which lie in Y. Note that δ-control may be lost during restriction, as the short paths in X between $f(v_i)$ and $f(v_j)$ may all leave Y. Assume that P restricts to Y, and let $Y' \subset Y$ be a further subspace. The theorem below shows that truncation may be used to restore control lost during restriction.

We call a linear map $P' : V \to V$ an ϵ-truncation of $P\|_Y : V \to V$ *over* Y' if (a) P' is ϵ-controlled w.r.t. the control map f and (b) there is some injection q (perhaps the identity) of the generators which map to Y' into all the generators $\{v_1, \ldots, v_a\}$, $q : f^{-1}(Y') \to f^{-1}(Y)$, satisfying $\mathrm{dist}_Y(v, q(v)) < \epsilon$ and such that if Q is the linear extension of q and W the span of $f^{-1}(Y')$, then

$$P'|_W = P \circ Q|_W.$$

[1] This assumption imposes very little limitation on the usefulness of the theory (see [7]).

TABLE 1
Truncated Daubechies Wavelet Matrices [a]

		$N = 2$	
1.00000000			
	0.86603650	0.48286291	0.12940952
	−0.50000000	0.83651630	0.22414387
		0.22414387	−0.836516630
		−0.12940952	0.48296291
0.48296291	0.12940952		
0.83651630	0.22414387		
0.22414387	−0.83651630	0.50000000	
−0.12940952	0.48296291	0.86602540	
			1.00000000
		$N = 3$	
0.92450814	−0.18268585	0.33267055	−0.03522629
−0.38116231	−0.44310402	0.80689151	−0.08544127
	0.81140256	0.45987750	0.13501102
	−0.33453040	−0.13501102	0.45987750
		−0.08544127	−0.80689151
		0.03522629	0.33267055
0.33267055	−0.03522629		
0.80689151	−0.08544127		
0.45987750	0.13501102	0.33453040	
−0.13501102	0.45987750	0.81140256	
−0.08544147	−0.80689151	0.44310402	0.37116231
0.03522629	0.33267055	−0.18268585	0.92450814

$N = 4$

1.00000000					
	0.95179333	−0.19320280	0.05979872	0.23037781	0.01059740
	−0.30674004	−0.59949507	0.18555135	0.71484657	0.03288301
		0.73926735	−0.03654874	0.63088077	−0.03084138
		−0.23824804	−0.62879714	−0.02798377	−0.18703481
			0.71560248	−0.18703481	0.02798377
			−0.23062143	0.03084138	0.63088077
				0.03288301	−0.71484657
				−0.01059740	0.23037781
0.23037781	0.01059740				
0.71484657	0.03288301				
0.63088077	−0.03084138	0.23062143			
−0.02798377	−0.18703481	0.71560248			
−0.18703481	0.02798377	0.62879714	0.23824804		
0.03084138	0.63088077	−0.03654874	0.73926735		
0.03288301	−0.71484657	−0.18555135	0.59949507	0.30674004	
−0.01059740	0.23037781	0.05979872	−0.19320280	0.95179333	
					1.00000000

TABLE 1
Truncated Daubechies Wavelet Matrices [a] (*Continued*)

$N = 5$

0.96660011	0.18674885	−0.06863167	0.02132779	0.16010240	−0.00333573
−0.25628933	0.70432688	−0.25884565	0.08043818	0.60382927	−0.01258075
	−0.66199655	−0.10351582	0.02719077	0.72430853	0.00624149
	0.17552517	0.72127384	−0.24291399	0.13842815	0.07757149
		−0.60929333	−0.13678226	−0.24229489	0.03224487
		0.16155117	0.72402136	−0.03224487	−0.24229489
			−0.60396032	0.07757149	0.13842815
			0.16013714	−0.00624149	0.72430853
				−0.01258075	−0.60382927
				0.00333573	0.16010240
0.16010240	−0.00333573				
0.60382927	−0.01258075				
0.72430853	0.00624149	0.16013714			
0.13842815	0.07757149	0.60396032			
−0.24229489	0.3224487	0.72402136	0.16155117		
−0.03224487	−0.24229489	0.13678226	0.60929333		
0.07757149	−0.13842815	−0.24291399	0.72127384	0.17552517	
−0.00624149	0.72430853	−0.02719077	0.10351582	0.66199655	
−0.01258075	−0.60382927	0.08043818	−0.25884565	0.70432688	0.25628933
0.00333573	0.16010240	−0.02132779	0.06863167	−0.18674885	0.96660011

$N = 6$

1.00000000							
	0.97550390	0.17477628	−0.06784932	0.02717362	−0.00780820	0.11154074	0.00107730
	−0.21998216	0.77503985	−0.30087567	0.12050058	−0.03462523	0.49462389	0.00477726
		−0.59238592	−0.24309190	0.11268704	−0.02533634	0.75113391	−0.00055384
		0.13358669	0.75639927	−0.23496233	0.09875033	0.31525035	−0.03158204
			−0.51028905	−0.30726624	0.12881915	−0.22626469	−0.02752287
			0.11507333	0.75102930	−0.22651996	−0.12976687	0.09750161
				−0.49585736	−0.31493063	0.09750161	0.12976687
				0.11181890	0.75109353	0.02752287	−0.22626469
					−0.49464696	−0.03158204	−0.31525035
					0.11154595	0.00055384	0.75113391
						0.00477726	−0.49462389
						−0.00107730	0.11154074

0.11154074	0.00107730						
0.49462389	0.00477726						
0.75113391	−0.00055384	0.11154505					
0.31525035	−0.03158204	0.49464696					
−0.22626469	−0.02752287	0.75109353	0.11181890				
−0.12976687	0.09750161	0.31493063	0.49585736				
0.09750161	0.12976687	−0.22651996	0.75102930	0.11507333			
0.02752287	−0.22626469	−0.12881915	0.30726624	0.51028905			
−0.03158204	−0.31525035	0.09875033	−0.23496233	0.75639927	0.13358669		
0.00055384	0.75113391	0.02533634	−0.11268704	0.24309190	0.59238592		
0.00477726	−0.49462389	−0.03462523	0.12050058	−0.30087567	0.77503985	0.21998216	
−0.00107730	0.11154074	0.00780820	−0.02717362	0.06784932	−0.17477628	0.97550390	
							1.00000000

[a] For each value of N, two triangular matrices are listed. Zero elements are denoted by blanks. Each matrix is column orthogonal. In the first (second) matrix of each pair, the two columns to the right (left) of vertical bars each contain a complete Daubechies coefficient vector. All the columns of each matrix are also orthogonal to all shifts down (up) of these two complete columns by 2, 4, ...

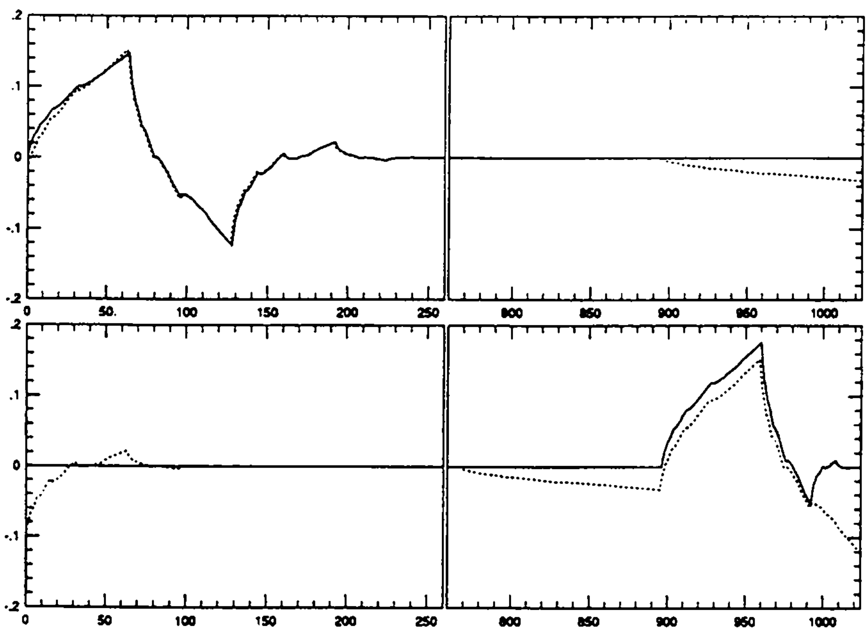


FIG. 4. Truncated and untruncated wavelets. Shown as solid curves are the leftmost (top) and rightmost (bottom) wavelet of one hierarchical scale, constructed on a vector of length 1024 using the truncated coefficient matrix developed in this paper (note break in the horizontal scale). The dotted curves, which wrap around, are the corresponding conventional wavelets, constructed with periodic boundary conditions.

The truncation is "over" Y' in the sense that P is basically unchanged on the generators mapping to Y'. The complexity of (b) results from a problem called *flux* (see Chapter 8 of [7]).

Often Q will be the identity, and then condition (b) is simply $P'|_W = P|_W$. To appreciate the role of Q in the general case (where flux may occur), consider the following example: X is the unit circle, and $f:\{v_1, \dots, v_p\} \to X$ is defined by $f(v_k) = e^{2\pi ik/p}$. Set $Y = X - \{e^{\pi i/p}\}$, $Y' = X \setminus I$, where I is an arc of the circle of length $2\pi/p$ centered at $e^{\pi i/p}$. Let P cyclically permute the v's, $P(v_k) = v_{k+1} \bmod p$. The P is clearly a $2\pi/p$-small orthogonal transformation. And according to our definitions, the identity $V \to V$ is a $2\pi/p$-truncation of $P\|_Y$ over Y'. But to find such a nonsingular truncation, we need the freedom to choose Q to cycle the generators mapping to Y' counterclockwise one step. Without a nontrivial Q there is no nonsingular extension of $P|_W$ which does not send the f-image of v_p across

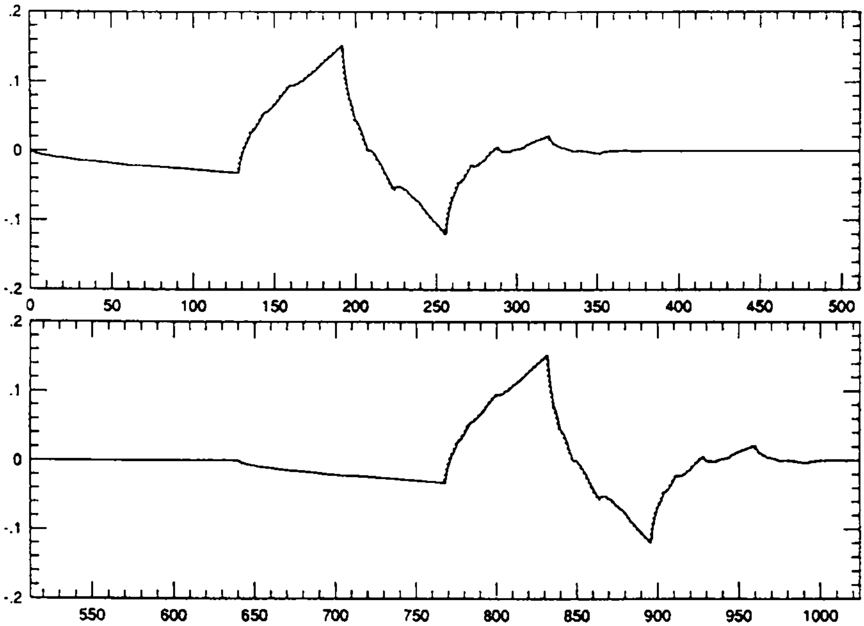


FIG. 5. Same as Figure 4, but for the second from left (top) and second from right (bottom) wavelets. One sees that the truncated and periodic wavelets are virtually identical. (In the continuum limit they would be identical.) The small difference comes from the offset by one of the rows of Daubechies coefficients in the interior of the matrix (see Table 1). Note that the distortion of the truncated wavelet as it nears the boundary will look different for different hierarchical scales.

the deleted points $e^{\pi i/p}$ and thus move it a great distance, $(1 - p)2\pi$, in the path metric on Y.

THEOREM. *For every $\epsilon > 0$ there exists a $\delta(D^k \to X) > 0$, depending on the geometry of X and the inclusion of its top cell, such that given any orthogonal transformation $O : V \to V$ of a finite dimensional inner product space V which is δ-controlled over X, the restriction $O\|_{D^k} : V \to V$ may be ϵ-truncated over $D' = D^k \setminus$ (a δ-neighborhood of ∂D^k) to an ϵ-controlled orthogonal transformation $O' : V \to V$ over D^k.*

Proof. We consider the case of X a torus with hexagonal "top cell" or "fundamental domain" D. We may form X as $R^2/\{$integral linear combinations of vectors p and q indicated in Figure 6$\}$.

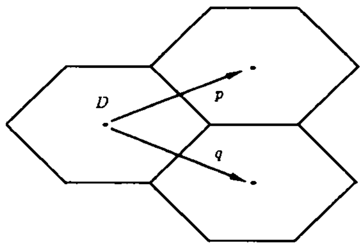


FIG. 6.

Let F denote frontier $D = \overline{D} \setminus D$. Cover F with round disks alternating between black and white (B_k and W_k) with diameter ϵ so that no two disks of the same color meet and so that the concentric subdisks B'_k and W'_k of diameter $\epsilon - 2\delta$ cover a δ-neighborhood of F as indicated in Figure 7.

The transformation $O : V \to V$ is δ-controlled over X. Let $V_0 \subset V$ be the subspace generated by those generators mapping outside a δ-neighborhood of the frontier, $f^{-1}(X \setminus \mathcal{N}_\delta \mathcal{F})) = \{v_0\}$, and let $C = O(V_0) \subset V$ be the "central" subspace. C is spanned by the orthonormal frame $\{Qv_0\} = \mathcal{F}$. Our task is to extend this frame of C to an orthonormal frame of V subject to the condition that for any additional frame vector w, the subset of $\{v_1, \ldots, v_a\}$ for which W has a nonzero component has radius $< \epsilon$ when mapped by f into D.

Consider an arbitrary black B_k. Dropping the index, B meets two or (in two cases) three widely separated regions of D (see Figure 4). If $v \in f^{-1}(B)$, we may correspondingly write $O(v) = \sum_{i=1}^{2 \text{ or } 3} z_i$, that is, z_i belongs to the span of generators which lie in B and one of the two or three distinct regions

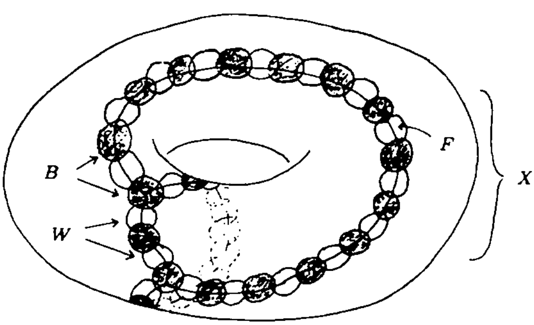


FIG. 7.

of D. Similarly if $v \in W$, a white ball, we may write $O(v) = y_1 + y_2$. Fixing B and i, we may apply the Gram-Schmidt procedure to produce from $\{z_i\}$ an orthonormal frame (of some indeterminant cardinality) $\mathcal{F}_B^i$ orthogonal to C. Since span$\{z_i\}$ is orthogonal to span$\{z_j\}$, $i \neq j = 1, 2, 3$, these frames are orthogonal for $i \neq j$. Furthermore, the disjointness of the black disks in Figure 2 assures that all these frames are disjoint as we vary B. Notice that the new frame vectors in $\mathcal{F}_B^i$ have diameter $< \epsilon$ in the sense that their nonzero components are always attached to vectors which map by f to a single B.

Now work with a single white disk W. Let $v \in f^{-1}(W')$ and $Ov = y_1 + y_2$ as before. For W and i fixed, $i = 1$ or 2, collect the fragments $\{y_i's\}$, and orthonormalize to produce an orthogonal frame orthogonal to span$\{\mathcal{F}, \mathcal{F}_B^i$, all B and $i\}$. Call the result $\mathcal{F}_W^i$. For the same reasons as before, the various $\mathcal{F}_W^i$ are themselves orthogonal, and their vector constituents have diameter $< \epsilon$. Thus $\bar{\mathcal{F}} = \{\mathcal{F}, \mathcal{F}_B^i, \mathcal{F}_W^i\}$ is an orthonormal frame. But clearly $\bar{\mathcal{F}}$ spans all of V, since it is made by applying Gram-Schmidt to a collection of vectors $O\{f^{-1}(X \setminus \mathcal{N}_\delta(F))\} \cup \{z_i's\} \cup \{y_i's\}$, which must be a spanning set, since it comes from the basis $O\{v_i\}$ on breaking several vectors into two or three summands. The point is that if a vector in a spanning set is broken into two or more summands and these replace the original vector in the spanning set, then the span cannot decrease.

The *new* vectors in the orthonormal basis $\bar{\mathcal{F}}$, i.e., those not in $\mathcal{F}$, have diameter $< \epsilon$; those in $\mathcal{F}$ have diameter $\leqslant \delta < \epsilon$. We may use a well-known combinatorial lemma [3], the marriage theorem, to redefine O in accordance with the theorem.

MARRIAGE THEOREM. *Suppose $\{b_i\}$ and $\{g_i\}$ are sets of equal cardinality and index set I, and that R is a relation such that for all $S \subset I$,* card$\{g_j$ *: for some $i \in S$, $b_i R g_j\}$* $\geqslant$ card(S). *It follows that there exists a bijection $h: I \to I$ such that for all $i \in I$, $b_i \, R \, g_{h(i)}$.*

Consider the relation $v_i R w_j$ that holds whenever w_j is an element of $\bar{\mathcal{F}}$ with a nonzero component in the coordinate direction $v_i \in \{v_1, \ldots, v_n\}$. Since $\bar{\mathcal{F}}$ is a frame, no subset of $\bar{\mathcal{F}}$ of cardinality c can be contained in the span of fewer than c basis elements. Thus the hypothesis of the marriage theorem holds. The bijection $v_i \to w_{h(i)}$ extends to an orthogonal transformation $O': V \to V$. Fixing j, the sets $f\{v_i$: nonzero coefficient for $w_j\}$ have diameter $\leqslant \epsilon$, so O' is ϵ-controlled. Furthermore, O' has the required form on $f^{-1}(X \setminus \mathcal{N}_\delta(F))$, $O' = O \circ Q$. Notice that Q might not be the identity, since a $v_0 \in f^{-1}(X \setminus N_\delta(F))$ might be "married" to a frame vector different from (but ϵ-close to) $O(v_0)$. Thus the use of the marriage theorem automatically accommodates the flux problem. This completes the theorem's proof in the case that X is a 2-torus with hexagonal top cell D.

In this example, the optimal relation between ϵ and δ is an exercise in plane geometry. For domains $D^2 \subset X$ with 120° angles and δ sufficiently small, $\epsilon \geqslant 10\delta$ is certainly adequate. In general ϵ will increase as the interior angles of D decrease (so regular hexagons are more efficient than squares in that ϵ can be chosen to be a smaller multiple of δ).

If X^k is a general Riemannian manifold, the first step is to locate a top cell D^k. Then one restricts to D^k and truncates to restore ϵ-control. In the general case, k distinct colors of balls must be chosen to cover the frontier of D^k. The vector $O(v)$ will be broken into at most $k + 1$ summands, collected into sets (fixing two indices—one labeling the ball, and the other the region of the top cell in which the summands' components land) and orthonormalized; then we inductively extend the central frame $\mathscr{F} = \{O(v_i) : f(v_i) \notin N_\delta(\text{frontier(top cell)})\}$. In general, $\epsilon(\delta)$ will increase with the dimension k of X^k and be sensitive to the geometry of X^k and the choice of the top cell D^k. ■

Let us review the truncation of the discrete Daubechies transforms, $M_{2n} \to M'_{2n}$, to see how it is a special case of the theorem. From the form of M_{2n} [see (3)] it can be regarded as an δ-small linear transformation for $\delta = 2\pi n/2^N$. To do this, let $(v_1, \ldots, v_{2^N})$ be the basis for the space V on which M_{2n} acts. Let $f(v_k) = e^{2\pi i k/2^N} \subset X = S^1$, the unit circle. While a band-diagonal form would suggest control over an interval, the shaded pattern in (3) corresponds to control over a circle:

$$(M_{2n})_{i,j} = 0 \qquad -n \leqslant i - j \leqslant n \bmod 2^N.$$

We cycled the original M_{2n} (1) into the form (3) precisely so that we would not have the notational inconveniences of flux (that is, the relabeling caused by Q). Proceeding from (3), we reduced the control space to an interval, $M_{2n} \to M_{2n}\|_{S^1 - \{0\}}$, and then truncated to restore lost control. That is, truncation removes all far from diagonal entries (M'_{2n} is n-band diagonal) while retaining M_{2n} [as in (3)] except for a modification of the top and bottom n rows. This means that if $X = S^1$ and $Y = S^1 - \{0\}$, then the space Y' over which M_{2n} is retrained is

$$Y' = S^1 - \{\text{the interval of length } \pi n/2^N \text{ centered at } 0\}.$$

Decomposing the top and bottom rows of M_{2n} as $v_i^{t,l} + v_i^{t,r}$ and $v_i^{b,l} + v_i^{b,r}$ respectively corresponds in our proof to $O(v) = y_1 + y_2$. The inner product structure on V is Euclidean in the preferred basis in which M_{2n} is written.

A special feature of this truncation is that no degradation of the data occurred. The matrix M'_{2n} is n-band diagonal where n is the radius of the diagonal band in M_{2n}. In terms of our theorem, this means $\epsilon = \delta$. This is certainly too much to hope for in general, but it would be interesting to see if there are other cases where the theorem holds with $\epsilon = \delta$.

REFERENCES

1 A. Cohen, I. Daubechies, B. Jaworth, and P. Vial, Multiresolution analysis, wavelets and fast algorithms on an interval, *C. R. Acad. Sci. Paris Ser. I Math.* 316:417–421 (1992).

2 Ingrid Daubechies, Orthonormal bases of compactly supported wavelets, *Comm. Pure Appl. Math.* XLI:909–996 (1988).

3 L. Lovasz and M. D. Plummer, *Matching Theory*, North-Holland, Amsterdam, 1986.

4 Stephane G. Mallat, A theory for multiresolution signal decomposition: The wavelet representation, *IEEE Trans. Pattern Anal. and Machine Intell.* 7:674–693 (1989).

5 Y. Meyer, Ondelettes sur' intervalle, *Rev. Math. Iberoamericana* 7(2):115–133 (1991).

6 W. H. Press, S. A. Teckolsky, W. T. VeHering, and B. P. Flannery, *Numerical Recipes in C: The Art of Scientific Computing*, 2nd ed., Cambridge U.P., 1992.

7 Frank Quinn, Ends of maps, I, *Ann. of Math.* 110:275–331 (1979).

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