Mathematics. — *On the attraction between two perfectly conducting plates.* By H. B. G. CASIMIR.

(Communicated at the meeting of May 29, 1948.)

In a recent paper by POLDER and CASIMIR [1]) it is shown that the interaction between a perfectly conducting plate and an atom or molecule with a static polarizibility α is in the limit of large distances R given by

$$\delta E = -\frac{3}{8\pi} \hbar c \frac{\alpha}{R^4}$$

and that the interaction between two particles with static polarizibilities α_1 and α_2 is given in that limit by

$$\delta E = -\frac{23}{4\pi} \hbar c \frac{\alpha_1 \alpha_2}{R^7}.$$

These formulae are obtained by taking the usual VAN DER WAALS–LONDON forces as a starting point and correcting for retardation effects.

In a communication to the "Colloque sur la théorie de la liaison chimique" (Paris, 12—17 April, 1948) the present author was able to show that these expressions may also be derived through studying by means of classical electrodynamics the change of electromagnetic zero point energy. In this note we shall apply the same method to the interaction between two perfectly conducting plates.

Let us consider a cubic cavity of volume L^3 bounded by perfectly conducting walls and let a perfectly conducting square plate with side L be placed in this cavity parallel to the xy face and let us compare the situation in which this plate is at a small distance a from the xy face and the situation in which it is at a very large distance, say $L/2$. In both cases the expressions $\frac{1}{2} \Sigma \hbar\omega$ where the summation extends over all possible resonance frequencies of the cavities are divergent and devoid of physical meaning but the *difference* between these sums in the two situations, $\frac{1}{2} (\Sigma \hbar \omega)_I - \frac{1}{2} (\Sigma \hbar \omega)_{II}$, will be shown to have a well defined value and this value will be interpreted as the interaction between the plate and the xy face.

The possible vibrations of a cavity defined by

$$0 \leqslant x \leqslant L, \quad 0 \leqslant y \leqslant L, \quad 0 \leqslant z \leqslant a$$

have wave numbers

$$k_x = \frac{\pi}{L} n_x, \quad k_y = \frac{\pi}{L} n_y, \quad k_z = \frac{\pi}{a} n_z,$$

where n_x, n_y, n_z are positive integers;

$$k = \sqrt{k_x^2 + k_y^2 + k_z^2} = \sqrt{\varkappa^2 + k_z^2}.$$

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[1]) H. B. G. CASIMIR and D. POLDER, Phys. Rev., **73**, 360 (1948).

To every k_x, k_y, k_z correspond *two* standing waves unless one of the n_i is zero, when there is only one. For k_x, k_y this is without importance since for very large L we may regard k_x, k_y as continuous variables. Thus we find

$$\tfrac{1}{2}\Sigma\hbar\omega=\hbar c\frac{L^2}{\pi^2}\int_0^\infty\int_0^\infty\left[\tfrac{1}{2}\sqrt{k_x^2+k_y^2}+\sum_{n=1}^{\infty}\sqrt{n^2\frac{\pi^2}{a^2}+k_x^2+k_y^2}\right]dk_x\,dk_y$$

or, introducing polar coordinates in the $k_x\,k_y$ plane,

$$\tfrac{1}{2}\Sigma\hbar\omega=\hbar c\frac{L^2}{\pi^2}\cdot\frac{\pi}{2}\sum_{(0)1}^{\infty}\int_0^\infty\sqrt{\left(n^2\frac{\pi^2}{a^2}+\varkappa^2\right)}\,\varkappa\,d\varkappa,$$

where the notation (0)1 is meant to indicate that the term with $n=0$ has to be multiplied by $\tfrac{1}{2}$. For very large a also this last summation may be replaced by an integral and it is therefore easily seen that our interaction energy is given by

$$\delta E=\hbar c\frac{L^2}{\pi^2}\cdot\frac{\pi}{2}\left\{\sum_{(0)1}^{\infty}\int_0^\infty\sqrt{\left(n^2\frac{\pi^2}{a^2}+\varkappa^2\right)}\,\varkappa\,d\varkappa-\int_0^\infty\int_0^\infty\sqrt{(k_z^2+\varkappa^2)}\,\varkappa\,d\varkappa\left(\frac{a}{\pi}dk_z\right)\right\}.$$

In order to obtain a finite result it is necessary to multiply the integrands by a function $f(k/k_m)$ which is unity for $k \ll k_m$ but tends to zero sufficiently rapidly for $(k/k_m)\to\infty$, where k_m may be defined by $f(1)=\tfrac{1}{2}$. The physical meaning is obvious: for very short waves (X-rays e.g.) our plate is hardly an obstacle at all and therefore the zero point energy of these waves will not be influenced by the position of this plate.

Introducing the variable $u=a^2\varkappa^2/\pi^2$,

$$\delta E=L^2\hbar c\frac{\pi^2}{4a^3}\left\{\sum_{(0)1}^{\infty}\int_0^\infty\sqrt{n^2+u}\,f(\pi\sqrt{n^2+u}/a\,k_m)\,du\right.$$
$$\left.-\int_0^\infty\int_0^\infty\sqrt{n^2+u}\,f(\pi\sqrt{n^2+u}/a\,k_m)\,du\,dn\right\}.$$

We apply the EULER–MACLAURIN formula:

$$\sum_{(0)1}^{\infty}F(n)-\int_0^\infty F(n)\,dn=-\tfrac{1}{12}F'(0)+\tfrac{1}{24\times30}F'''(0)+\dots$$

Introducing $w=u+n^2$ we have

$$F(n)=\int_{n^2}^{\infty}w^{1/2}f(w\pi/a\,k_m)\,dw,$$

whence

$$F'(n)=-2n^2f(n^2\pi/a\,k_m)$$
$$F'(0)=0$$
$$F'''(0)=-4.$$

The higher derivatives will contain powers of (π/ak_m). Thus we find

$$\delta E/L^2 = -\hbar c \frac{\pi^2}{24 \times 30} \cdot \frac{1}{a^3},$$

a formula which holds as long as $ak_m \gg 1$. For the force per cm^2 we find

$$F = \hbar c \frac{\pi^2}{240} \frac{1}{a^4} = 0{,}013 \frac{1}{a_\mu^4} \text{ dyne/cm}^2$$

where a_μ is the distance measured in microns.

We are thus led to the following conclusions. There exists an attractive force between two metal plates which is independent of the material of the plates as long as the distance is so large that for wave lengths comparable with that distance the penetration depth is small compared with the distance. This force may be interpreted as a zero point pressure of electromagnetic waves.

Although the effect is small, an experimental confirmation seems not unfeasable and might be of a certain interest.

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