GRAPH ISOMORPHISM AND REPRESENTATION THEORY

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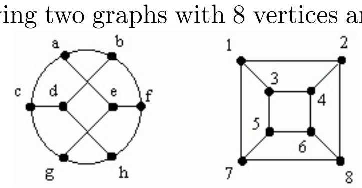
1. Introduction

How many isomorphism classes of (unlabeled) graphs are there with n vertices and k edges? (Here a graph is a set V of vertices, and a set E of (unordered) pairs of distinct elements of V, which we think of as edges between two vertices. In particular, we do not allow loops or multiple edges between two vertices. We say two graphs $(V_1, E_1), (V_2, E_2)$ are isomorphic if there is a bijection $V_1 \simeq V_2$ sending E_1 bijectively to E_2.)

For example, the following two graphs with 8 vertices and 12 edges are isomorphic.



Below we record the number of graphs with n vertices, as the number of edges range from 0 to $\binom{n}{2}$:

n	# of graphs with n vertices and $0, 1, 2, \cdots$ edges
1	1
2	1, 1
3	1, 1, 1, 1
4	1, 1, 2, 3, 2, 1, 1
5	1, 1, 2, 4, 6, 6, 6, 4, 2, 1, 1
6	1, 1, 2, 5, 9, 15, 21, 24, 24, 21, 15, 9, 5, 2, 1, 1
7	1, 1, 2, 5, 10, 21, 41, 65, 97, 131, 148, 148, 131, 97, 65, 41, 21, 10, 5, 2, 1, 1

Let $g_{n,k}$ be the number of isomorphism classes of (unlabeled) graphs with n vertices and k edges.

Observe: each row of the table above is

(1) Symmetric (i.e. $g_{n,k} = g_{n,\binom{n}{2}-k}$ for all k, n), and
(2) Unimodal (i.e. $g_{n,k}$ increases, and then decreases).

Proof of (1). There is a bijection between (unlabeled) graphs with k edges and graphs with $\binom{n}{2} - k$ edges, given by sending a graph to its complement. □

The rest of this talk will be devoted to a proof of (2), that is, that the sequence $\{g_{n,k}\}$ for fixed n is unimodal. The idea of the proof is to turn this into a problem of linear algebra — in particular, we will use the representation theory of the Lie algebra $\mathfrak{sl}_2$.

2. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$

As a vector space, we define

$$\mathfrak{sl}_2(\mathbb{C}) = \{A \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \mid \mathrm{Tr}(A) = 0\}.$$

That is, $\mathfrak{sl}_2(\mathbb{C})$ consists of the set of 2×2 traceless matrices. The product of two traceless matrices need not be traceless, so this space does not have a natural product. That said, the commutator of two traceless matrices

$$[A, B] = AB - BA$$

is traceless. We view $\mathfrak{sl}_2(\mathbb{C})$ as the above vector space, with this commutator as a binary operation. The commutator (which we call the *Lie bracket* from now on) is bilinear and satisfies

(1) Anti-commutativity:
$$[A, B] = -[B, A],$$
(2) The Jacobi identity:
$$[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0.$$

A vector space V with a bilinear map

$$[-, -] : V \times V \to V$$

satisfying the above properties is called a *Lie algebra.*

Let us describe $\mathfrak{sl}_2(\mathbb{C})$ in terms of a basis. We set

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The set $\{e, f, h\}$ is evidently a basis for $\mathfrak{sl}_2(\mathbb{C})$. A short computation gives

$$[e, f] = h, [h, f] = -2f, [h, e] = 2e,$$

which determines the Lie bracket in general by bilinearity and anti-commutativity.

Definition 2.1. A representation of a Lie algebra $\mathfrak{g}$ is a linear map

$$\rho : \mathfrak{g} \to \mathfrak{gl}_n(\mathbb{C}) := \mathrm{Mat}_{n\times n}(\mathbb{C})$$

such that

$$\rho([A, B]) = [\rho(A), \rho(B)]$$

for all A, B.

In particular, an n-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ is a set of three $n \times n$ matrices E, F, H such that

$$[E, F] = H, [H, F] = -2F, [H, E] = 2E.$$

3. Representation theory of $\mathfrak{sl}_2(\mathbb{C})$

We now describe explicitly the finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$. Let V be a finite-dimensional $\mathbb{C}$-vector space and $E, F, H : V \to V$ linear maps satisfying the commutator relations above.

Theorem 3.1. *E, F are nilpotent. If v is an eigenvector of H with eigenvalue λ, then $H(Ev) = (\lambda + 2)Ev$; if Fv is non-zero, $H(Fv) = (\lambda - 2)Fv$.*

Proof. Let v be an eigenvector of H with eigenvalue λ, i.e.

$$Hv = \lambda v.$$

Then

$$HEv = 2Ev + EHv = (\lambda + 2)Ev$$

and

$$HFv = FHv - 2Fv = (\lambda - 2)Fv$$

as desired. So $\lambda+2, \lambda-2$ are both eigenvalues of H, and Fv, Ev are both eigenvectors (if they are non-zero). Hence if E is not nilpotent, we have that $\lambda+2n$ is an eigenvalue for all positive n; likewise if F is not nilpotent, $\lambda-2n$ is an eigenvalue for all positive n. But H has only finitely many eigenvalues (as V is finite-dimensional), so E, F act nilpotently on any eigenvector of H, and hence on the subspace V^{ss} spanned by the eigenvectors of H. By the computations above, E, F preserve this subspace, so we may replace V by V/V^{ss}, whence we are done by induction on the dimension of V. □

Theorem 3.2. *Let λ be an eigenvalue of H with maximal real part, and let v be a λ-eigenvector. Let N be minimal such that $F^N v = 0$. Then $Ev = 0$, and* $\mathrm{Span}(v, Fv, F^2v, \cdots, F^{N-1}v)$ *is a subrepresentation of V. Moreover $\lambda = N - 1$.*

Proof. $Ev = 0$ as if not, $\lambda + 2$ would be an eigenvalue of H, contradicting the maximality of the real part of λ. Then

$$\lambda v = Hv = [E, F]v = EFv - FEv = EFv,$$

$$(\lambda - 2)Fv = HFv = [E, F]Fv = EF^2v - FEFv$$

and hence

$$EF^2v = (\lambda - 2)Fv + F\lambda v = (2\lambda - 2)Fv$$

and in general

$$(\lambda - 2(n-1))F^{n-1}v = HF^{n-1}v = [E, F]F^{n-1}v = EF^n v - FEF^{n-1}v$$

and so

$$EF^n v = (\lambda - 2(n-1))F^{n-1}v + FEF^{n-1}v = (-n^2 + (\lambda + 1)n)F^{n-1}v$$

by induction on n (exercise). Thus E sends $\mathrm{Span}(F^n v)$ to $\mathrm{Span}(F^{n-1}(v))$, F sends $\mathrm{Span}(F^{n-1}v)$ to $\mathrm{Span}(F^n v)$, and H preserves $\mathrm{Span}(F^n v)$, so the vector space in question is a subrepresentation as desired.

Now take $n = N$, so that $F^N v = 0$ and $F^{N-1} \neq 0$. Then we have

$$0 = EF^N v = (-N^2 + (\lambda + 1)N)F^{N-1}v$$

and hence

$$-N^2 + (\lambda + 1)N = 0.$$

Thus

$$\lambda = N - 1$$

as desired. □

In fact the proof gives an explicit description of all the irreducible finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$. Given any irreducible representation V, we may take λ with maximal real part as above — then the subrepresentation given by the theorem must be all of V by irreducibility.

Indeed, the proof shows that there is a unique irreducible $\mathfrak{sl}_2(\mathbb{C})$-representation V_N of dimension N, which we may describe explicitly as follows. Namely V_N has a basis $\{v_0, v_1, \cdots, v_{N-1}\}$ so that

$$Fv_n = v_{n+1} \text{ for } n < N-1 \text{ and } Fv_{N-1} = 0$$
$$Hv_n = (N-1-2n)v_n$$
$$Ev_n = (-n^2 + (\lambda+1)n)v_{n-1}.$$

(In fact, all finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$ are semisimple, i.e. they split up into direct sums of copies of the V_N above, so this gives a description of *all* finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$. But we will not use this fact.)

Remark 3.3. One may give a more "coordinate-free" description of the irreducible representations. Namely V_N may be viewed as the space of homogeneous polynomials in two variables X, Y of degree $N-1$. Then E acts via $X\frac{\partial}{\partial Y}$, F via $Y\frac{\partial}{\partial X}$, and H via $X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}$.

We record the key fact that these results give us about $\mathfrak{sl}_2(\mathbb{C})$-representations, for future use:

Theorem 3.4. *Let V be a finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-representation. Let d_k be the dimension of the generalized k-eigenspace of H acting on V. Then the sequences*

$$\{d_k\}_{k \text{ odd}}, \{d_k\}_{k \text{ even}}$$

are both unimodal and symmetric about 0 (i.e. $d_k = d_{-k}$ for all k).

Proof. The statement is true for irreducible representations by our classification, and is preserved under extensions (which have the effect of summing the sequences in question). Thus it is true for arbitrary representations. □

4. Back to graph theory

We now have the tools to return to our original problem. Recall that $g_{n,k}$ was the number of unlabeled graphs with n vertices and k edges; we wish to show that for fixed n, the sequence $g_{n,k}$ is unimodal. The idea will to be to construct an $\mathfrak{sl}_2$-representation such that the $g_{n,k}$ appear as dimensions of H-eigenspaces as in Theorem 3.4.

We fix n, and let W_n be the vector space on the set of *labelled* graphs on the vertices $\{1, \cdots, n\}$. That is, if G_n is the set of labelled graphs on the vertices $\{1, \cdots, N\}$,

$$W_n = \bigoplus_{g \in G_n} \mathbb{C}g.$$

The symmetric group S_n acts on G_n by permuting the labels of a labelled graph, and hence acts on W_n. We let $W_{n,k} \subset W_n$ be the subspace spanned by $g \in G_n$ such that g has k edges. Observe that $W_{n,k}$ is a subrepresentation.

Proposition 4.1.

$$g_{n,k} = \dim W_{n,k}^{S_n}.$$

Proof. A basis for $W_{n,k}^{S_n}$ is given by elements of the form

$$\sum_{\sigma \in S_n} \sigma(g)$$

where g is a labelled graph with n vertices and k edges. It is clear that this basis is in bijection with isomorphism classes of graphs with k edges. □

We now define an action of $\mathfrak{sl}_2(\mathbb{C})$ on W_n. Let $a_{i,j} : W \to W$ be the operator

$$a_{i,j} : g \mapsto \begin{cases} g \cup (i,j) & \text{if } (i,j) \notin g \\ 0 & \text{otherwise} \end{cases}$$

i.e. $a_{i,j}$ adds an edge to g between vertices i and j if there isn't one there already, and sends g to 0 otherwise. Let $b_{i,j}$ be the operator

$$b_{i,j} : g \mapsto \begin{cases} g \setminus (i,j) & \text{if } (i,j) \in g \\ 0 & \text{otherwise} \end{cases}$$

i.e. $b_{i,j}$ removes the edge between i and j if such an edge exists, and sends g to 0 otherwise. Note that if $\{s,t\} \neq \{u,v\}$,

$$[a_{s,t}, b_{u,v}] = 0,$$

and

$$[a_{s,t}, b_{s,t}]g = \begin{cases} g \text{ if } (s,t) \text{ is an edge in } g \\ -g \text{ otherwise} \end{cases}$$

We set

$$E = \sum_{i<j} a_{i,j}$$

and

$$F = \sum_{i<j} b_{i,j}.$$

Then for g a labelled graph with k edges,

$$\begin{aligned} [E,F](g) &= \left[\sum_{s<t} a_{s,t}, \sum_{u<v} b_{u,v} \right] g \\ &= \sum_{s<t, u<v} [a_{s,t}, b_{u,v}]g \\ &= \sum_{s<t} [a_{s,t}, b_{s,t}]g \\ &= \sum_{(s,t)\in g} g - \sum_{(s,t)\notin g} g \\ &= \left(2k - \binom{n}{2}\right) g. \end{aligned}$$

Thus we set $H_k : W_{n,k} \to W_{n,k}$ equal to the operator

$$H_k : g \mapsto \left(2k - \binom{n}{2}\right) g,$$

and

$$H = \bigoplus_k H_k.$$

We have already verified that $[E, F] = H$. Let us verify the other two commutation relations for $\mathfrak{sl}_2(\mathbb{C})$. Indeed, we have that for g a graph with k edges,

$$\begin{aligned}[H, F]g &= \left(2k - 2 - \binom{n}{2}\right) Fg - F\left(2k - \binom{n}{2}\right) g \\ &= -2Fg\end{aligned}$$

and

$$\begin{aligned}[H, E]g &= \left(2k + 2 - \binom{n}{2}\right) Eg - E\left(2k - \binom{n}{2}\right) g \\ &= 2Eg\end{aligned}$$

as desired.

Observe that the action of $\mathfrak{sl}_2(\mathbb{C})$ on W_n commutes with the action of S_n, and so we get an action on

$$W_n^{S_n} = \bigoplus_k W_{n,k}^{S_n}.$$

We are now ready to prove the theorem.

Theorem 4.2. *The sequence $\{g_{n,k}\}$ (for fixed n and varying k) is unimodal.*

Proof. We already know that $g_{n,k} = \dim W_{n,k}^{S_n}$. But we have already constructed an action of $\mathfrak{sl}_2(\mathbb{C})$ on

$$W_n^{S_n} = \bigoplus_k W_{n,k}^{S_n}$$

such that H acts diagonalizably, and the eigenspace of H corresponding to the eigenvalue $2k - \binom{n}{2}$ is precisely $W_{n,k}^{S_n}$. Thus we are done by Theorem 3.4, as the numbers $2k - \binom{n}{2}$ all have the same parity (namely, the parity of $\binom{n}{2}$). □

Remark 4.3. Note that the same argument proves more — namely that for every irreducible representation χ of S_n, the sequence

$$\dim W_{n,k}^{\chi}$$

is unimodal.

Remark 4.4. Note that this gives a fundamentally *different* proof that $g_{n,k} = g_{n,\binom{n}{2}-k}$.

Exercise 4.5. *Let $P_{kl}(n)$ be the number of ways of partitioning n into at most k pieces, each of which has size at most l. Show that for fixed k and l, the sequence*

$$P_{kl}(n)$$

is unimodal.

Question 4.6 (For experts only). In geometry, if X is a smooth projective variety, there are natural $\mathfrak{sl}_2$-representations on the vector space

$$H^*(X, \mathbb{C}),$$

associated to ample classes on X. This is the *hard Lefschetz theorem.* A natural question is — is there a natural variety whose cohomology is of the form $W_n^{S_n}$?

Even better, is there a natural variety with an S_n-action whose cohomology is W, viewed as an $S_n \times \mathfrak{sl}_2$-representation?