

Introduction to tropical algebraic geometry

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ABSTRACT. This is an expository introduction to tropical algebraic geometry based on my lectures at the Workshop on Tropical Geometry and Integrable Systems in Glasgow, July 4–8, 2011, and at the ELGA 2011 school on Algebraic Geometry and Applications in Buenos Aires, August 1–5, 2011.

1. Introduction

Tropical algebraic geometry is algebraic geometry over the tropical semiring (Definition 1.1). This replaces an algebraic variety by a piecewise linear object which can be studied using polyhedral combinatorics.

Tropical geometry has exploded as an area of research in the last decade, with many new connections and applications appearing each year. These include enumerative geometry, mirror symmetry, arithmetic geometry, and integrable systems. It builds on the older area of tropical mathematics, more commonly known as max-plus algebra, which arises in semigroup theory, computer science, and optimization. The name “tropical” was coined by some French mathematicians in honor of the Brazilian computer scientist Imre Simon. See [6] or [19] for an introduction to this older area.

The goal of this expository and elementary article is to introduce this exciting new area. We develop the theory of tropical varieties and outline their structure and connection with “classical” varieties.

There are several approaches to tropical geometry. We follow the “embedded” approach, which focuses on tropicalizing classical varieties. Another important branch of the subject focuses on developing an abstract theory of tropical varieties in their own right. See work of Mikhalkin and collaborators [29], [30] for details on this. This direction is most developed for curves. One of the attractions of tropical geometry is that it has so many disparate, but connected, facets, so any survey is necessarily incomplete.

We begin by introducing tropical mathematics, focusing on tropical polynomials and their solutions. In this paper we will follow the minimum convention for the tropical semiring:

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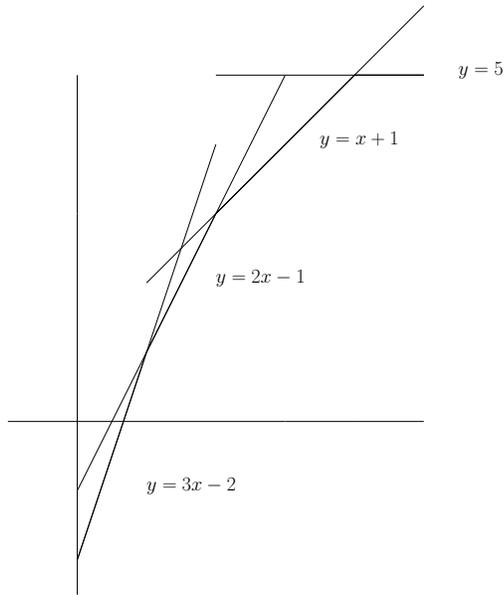


FIGURE 1. A tropical polynomial

DEFINITION 1.1. The *tropical semiring* is $\mathbb{R} \cup \{\infty\}$, with operation \oplus and \otimes given by $a \oplus b = \min(a, b)$ and $a \otimes b = a + b$.

The tropical semiring is associative and distributive, with additive identity ∞ and multiplicative identity 0. This satisfies every axiom of a ring except additive inverses, so is a semiring.

Tropical operations are often simpler than regular operations. For example we have the “Freshman’s dream”: $(x \oplus y)^n = x^n \oplus y^n$. The following examples illustrate that tropical polynomials are piecewise linear functions.

- EXAMPLE 1.2. (1) The tropical polynomial $F(x) = -2 \otimes x^3 \oplus -1 \otimes x^2 \oplus 1 \otimes x \oplus 5$ is $\min(3x - 2, 2x - 1, x + 1, 5)$ in regular arithmetic. This is the piecewise linear function whose graph is shown in Figure 1.
- (2) The tropical multivariate polynomial $x \oplus y \oplus 0$ is the piecewise linear function $\min(x, y, 0)$ in regular arithmetic. Note that the zero cannot be removed here, as zero is not the additive identity. This is a function from \mathbb{R}^2 to \mathbb{R} whose domain is shown in Figure 2.

With no subtraction, it is not obvious how to solve polynomial equations. For example, the equation $x \oplus 2 = 5$ has no solution. This problem has the following resolution.

DEFINITION 1.3. The *tropical hypersurface* $V(F)$ defined by the tropical polynomial F in n variables is the nondifferentiable locus of F in \mathbb{R}^n . This is the set of $x \in \mathbb{R}^n$ for which the minimum is achieved at least twice in $F(x)$.

EXAMPLE 1.4. For the first polynomial of Example 1.2, $V(F) = \{1, 2, 4\}$. For the second, $V(F)$ is the union of the three rays shown on the right in Figure 2.

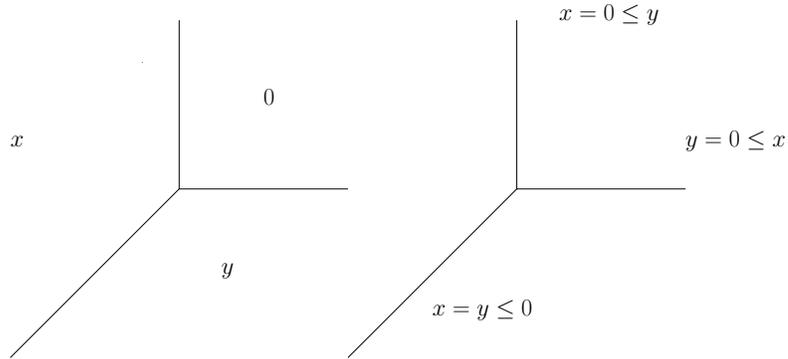


FIGURE 2. A tropical line

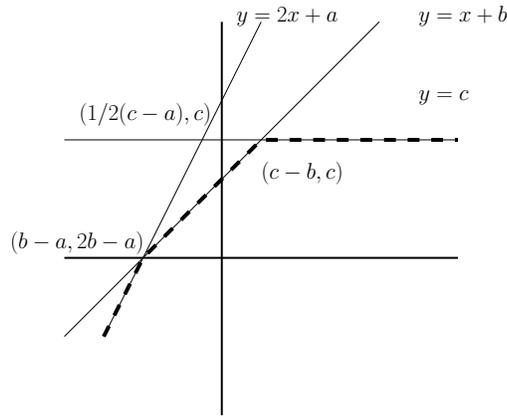


FIGURE 3. Tropical quadratic polynomials

EXAMPLE 1.5. The tropical quadratic formula is particularly simple. If $F(x) = a \otimes x^2 \oplus b \otimes x \oplus c$, then the graph of F is shown in Figure 3. Note that there are two cases, depending on the sign of the tropical discriminant $a + c - 2b$. If $2b \leq a + c$ then $V(F) = \{c - b, b - a\}$. If $2b \geq a + c$ then $V(F) = \{(c - a)/2\}$. Compare this with the usual quadratic formula, and the usual discriminant. Higher degree polynomials have similarly easy solutions.

2. Tropical varieties

The impact of tropical techniques comes when they are applied to “classical” objects, such as algebraic varieties. We now explain how to tropicalize certain algebraic varieties.

For a field K we set $K^* = K \setminus \{0\}$. Fix a valuation $\text{val} : K^* \rightarrow \mathbb{R}$. This is a function satisfying:

- (1) $\text{val}(ab) = \text{val}(a) + \text{val}(b)$
- (2) $\text{val}(a + b) \geq \min(\text{val}(a), \text{val}(b))$.

EXAMPLE 2.1. (1) $K = \mathbb{C}$ with the trivial valuation $\text{val}(a) = 0$ for all $a \in \mathbb{C}^*$.

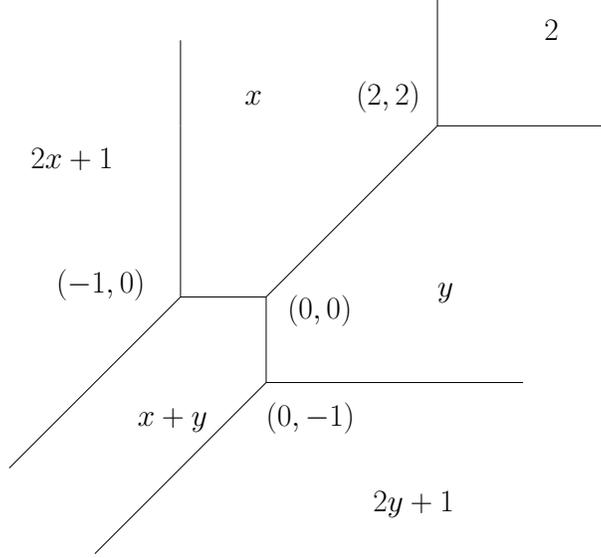


FIGURE 4. A tropical quadric

- (2) $K = \mathbb{C}\{\{t\}\}$, the field of Puiseux series. This is the union $\bigcup_{n \geq 1} \mathbb{C}((t^{1/n}))$, and is the algebraic closure of the field of Laurent series. Elements are Laurent series with rational exponents where in any given series the exponents all have a common denominator. The valuation of $a \in \mathbb{C}\{\{t\}\}$ is the lowest exponent appearing. For example, $\text{val}(3t^{-1/2} + 8t^2 + 7t^{13/3} + \dots) = -1/2$.
- (3) $K = \mathbb{Q}$ or \mathbb{Q}_p with the p -adic valuation. If $q = p^n a/b \in \mathbb{Q}$ with p not dividing a or b , then $\text{val}_p(q) = n$. For example, $\text{val}_2(8) = 3$, and $\text{val}_3(5/6) = -1$.

DEFINITION 2.2. The *tropicalization* of a Laurent polynomial $f = \sum c_u x^u \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is $\text{trop}(f) : \mathbb{R}^n \rightarrow \mathbb{R}$ given by

$$\text{trop}(f)(w) = \min(\text{val}(c_u) + w \cdot u).$$

This is obtained by regarding the addition and multiplication as tropical addition and multiplication, and changing the coefficients to their valuations.

EXAMPLE 2.3. Let $K = \mathbb{Q}$ with the 2-adic valuation, and let $f = 6x^2 + 5xy + 10y^2 + 3x - y + 4 \in \mathbb{Q}[x^{\pm 1}, y^{\pm 1}]$. Then $\text{trop}(f) = \min(2x + 1, x + y, 2y + 1, x, y, 2)$. This is illustrated in Figure 4.

For $f \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ the (classical) hypersurface $V(f)$ equals $\{x \in (K^*)^n : f(x) = 0\}$. The tropicalization $\text{trop}(V(f))$ of $V(f)$ is the tropical hypersurface of $\text{trop}(f)$. This is the nondifferentiable locus of $\text{trop}(f)$, or equivalently:

$$\text{trop}(V(f)) = \{w \in \mathbb{R}^n : \text{the minimum in } \text{trop}(f)(w) \text{ is achieved at least twice}\}.$$

Note that $\text{trop}(x^u f) = \text{trop}(x^u) + \text{trop}(f)$, so $\text{trop}(x^u f)(w) = w \cdot u + \text{trop}(f)(w)$, and thus $\text{trop}(V(x^u f)) = \text{trop}(V(f))$. This explains why the natural algebraic varieties to tropicalize are subvarieties of $T = (K^*)^n$, rather than subvarieties of \mathbb{A}^n or \mathbb{P}^n , as monomial functions are invertible on T .

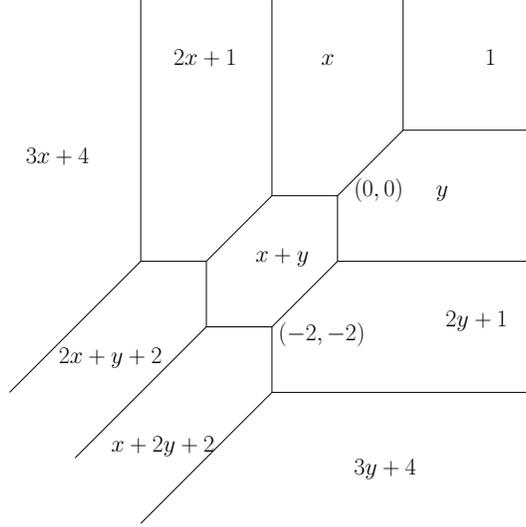


FIGURE 5. A tropical elliptic curve

Let $Y = V(I)$ be a subvariety of $T = (K^*)^n$. If $I = \langle f_1, \dots, f_r \rangle$ then

$$\begin{aligned} Y = V(I) &= \{x \in T : f_1(x) = \dots = f_r(x) = 0\} \\ &= \{x \in T : f(x) = 0 \text{ for all } f \in I\} \\ &= \bigcap_{f \in I} V(f). \end{aligned}$$

DEFINITION 2.4. The tropicalization of a variety $Y = V(I) \subseteq T$ is

$$\text{trop}(Y) = \bigcap_{f \in I} \text{trop}(V(f)).$$

EXAMPLE 2.5. (1) $Y = V(x + y + z + w, x + 2y + 5z + 11w) \subseteq (\mathbb{C}^*)^4$.

Then $\text{trop}(Y)$ has the property that if $\alpha \in \text{trop}(Y)$, then $\alpha + \lambda(1, 1, 1, 1) \in \text{trop}(Y)$ for all $\lambda \in \mathbb{R}$. We can thus quotient by the span of $(1, 1, 1, 1)$ and describe $\text{trop}(Y)$ in $\mathbb{R}^4 / \mathbb{R}(1, 1, 1, 1) \cong \mathbb{R}^3$. This consists of four rays, being the images of the positive coordinate directions $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$.

(2) Let $Y = V(t^3x^3 + x^2y + xy^2 + t^3y^3 + x^2 + t^{-1}xy + y^2 + x + y + t^3) \subseteq (\mathbb{C}\{\{t\}\}^*)^2$. Then $\text{trop}(V(f))$ is shown in Figure 5. This is a “tropical elliptic curve”; see, for example, [12] or [26].

REMARK 2.6. If $Y = V(f_1, \dots, f_r)$, the tropical variety $\text{trop}(Y)$ does not always equal $\bigcap_{i=1}^r \text{trop}(V(f_i))$. For example, in the first part of Example 2.5, $\text{trop}(V(x + y + z + w)) = \text{trop}(V(x + 2y + 5z + 11w))$, but this is larger than the tropical variety $\text{trop}(Y)$.

DEFINITION 2.7. If $Y = V(I)$, a set $\{f_1, \dots, f_r\} \subseteq I$ with

$$\text{trop}(Y) = \bigcap_{i=1}^r \text{trop}(V(f_i))$$

is called a *tropical basis* for I . Finite tropical bases always exist (see [5, Theorem 11]), so $\text{trop}(Y)$ is a piecewise linear object.

The following is the guiding question behind much tropical geometry research:

QUESTION 2.8. Which properties of Y or of compactifications of Y can be deduced from $\text{trop}(Y)$?

3. Drawing curves in the plane

Before delving deeper into the connections between classical and tropical varieties, with a view to partial answers to Question 2.8, we first describe how to draw plane curves.

Let $C = V(f) \subseteq (K^*)^2$ for $f \in K[x^{\pm 1}, y^{\pm 1}]$. If $f = \sum c_{ij} x^i y^j$, then $\text{trop}(f) = \min(\text{val}(c_{ij}) + ix + jy)$, and $\text{trop}(C) = \text{trop}(V(f))$ is the locus where the minimum is achieved at least twice. Some elementary polyhedral geometry gives a shortcut to compute this locus, so we first review this notation. An excellent general introduction to polyhedral geometry is [39].

DEFINITION 3.1. A *polyhedron* in \mathbb{R}^n is the intersection of finitely many half-spaces in \mathbb{R}^n . This can be written as:

$$P = \{x \in \mathbb{R}^n : Ax \leq b\},$$

where A is a $d \times n$ matrix, and $b \in \mathbb{R}^d$. The dimension of P is the dimension of the subspace $\ker(A)$. For a subgroup $\Gamma \subseteq \mathbb{R}$, we say that a polyhedron P is Γ -rational if A has rational entries, and $b \in \Gamma^d$. When $\Gamma = \mathbb{Q}$, we say that the polyhedron is rational.

If $b = 0$, then P is called a *cone*. In that case there are $\mathbf{v}_1, \dots, \mathbf{v}_s$ for which $P = \text{pos}(\mathbf{v}_1, \dots, \mathbf{v}_s) := \{\sum_{i=1}^s \lambda_i \mathbf{v}_i : \lambda_i \geq 0\}$.

The *face* of a polyhedron P determined by $w \in (\mathbb{R}^n)^*$ is the set

$$\text{face}_w(P) = \{x \in P : w \cdot x \leq w \cdot y \text{ for all } y \in P\}.$$

EXAMPLE 3.2. Let $P \subset \mathbb{R}^2$ be the square with vertices $\{(0, 0), (1, 0), (0, 1), (1, 1)\}$. This has the description

$$P = \left\{ x \in \mathbb{R}^2 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} x \leq \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Then we have

- (1) $\text{face}_{(1,0)}(P)$ is the edge of the square with vertices $\{(0, 0), (0, 1)\}$,
- (2) $\text{face}_{(1,1)}(P)$ is the vertex $(0, 0)$, and
- (3) $\text{face}_{(0,0)}(P)$ is P .

DEFINITION 3.3. A *polyhedral complex* Σ is a finite union of polyhedra for which any nonempty intersection of two polyhedra $\sigma_1, \sigma_2 \in \Sigma$ is a face of each. If every polyhedron in Σ is a cone, then Σ is called a fan.

The *normal fan* to a polyhedron P is the fan $N(P)$ with cones $C[w] = \text{cl}(w' : \text{face}_{w'}(P) = \text{face}_w(P))$, where $\text{cl}(\cdot)$ is the closure in the usual Euclidean topology on \mathbb{R}^n . This is sometimes called the inner normal fan, as the cones corresponding to facets (faces of maximal dimension) are the inner normal vectors to these faces

DEFINITION 3.4. Fix $f \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. The *Newton polytope* of $f = \sum c_u x^u$ is the convex hull of the $u \in \mathbb{Z}^n$ with $c_u \neq 0$: $P = \{\sum_{c_u \neq 0} \lambda_u u : \sum \lambda_u = 1\}$.

EXAMPLE 3.5. Let $f = x^2y + 5y^2 - 3x + 2$. The Newton polytope P of f and its normal fan $N(P)$ are illustrated in Figure 6. The polytope P is the convex hull of the exponent vectors $\{(2, 1), (0, 2), (1, 0), (0, 0)\}$.

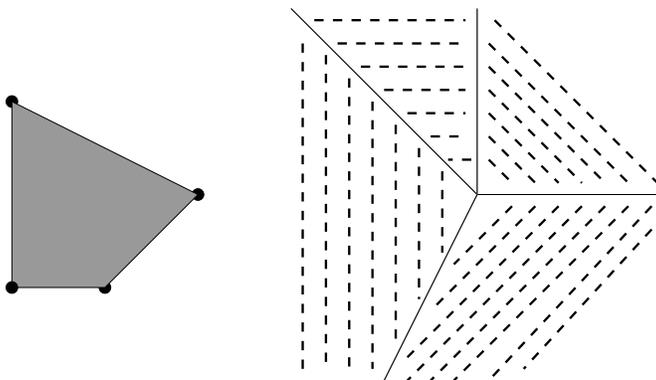


FIGURE 6. Newton polytope and normal fan

We now temporarily restrict our attention to plane curves, so $n = 2$. The tropicalization of a plane curve $C = V(f)$ for $f = \sum c_{ij} x^i y^j$ depends on the valuations of the coefficients c_{ij} . If the valuation on K is trivial ($\text{val}(a) = 0$ for all $a \neq 0$), then $\text{trop}(C)$ is the union of all one-dimensional cones in the normal fan $N(P)$ to the Newton polytope of f .

We now consider the case that K has a nontrivial valuation. Let \tilde{P} be the convex hull of the set $\{(i, j, \text{val}(c_{ij})) : c_{ij} \neq 0\}$ in $\mathbb{R}^{2+1} = \mathbb{R}^3$, and let $N(\tilde{P})$ be its normal fan. The regular subdivision $\Delta_{(\text{val}(c_{ij}))}$ of P corresponding to the vector $(\text{val}(c_{ij}))$ is the projection to P of the “lower faces” of \tilde{P} . More information about regular subdivisions can be found in [20, Chapter 7] and [11, Chapter 2].

EXAMPLE 3.6. Let $f = 2x^2 + xy - 6y^2 + 5x - 3y + 2 \in \mathbb{Q}[x^{\pm 1}, y^{\pm 1}]$ where \mathbb{Q} has the 2-adic valuation. Then the regular subdivision of the Newton polytope of f corresponding to $\text{val}(c_u)$ is shown in Figure 7.

The tropicalization of C is $\text{trop}(C) = \{w \in \mathbb{R}^2 : \text{face}_{(w,1)}(\tilde{P}) \text{ is not a vertex}\}$. This is the reflection of the dual graph to $\Delta_{(\text{val}(c_{ij}))}$ under $x \mapsto -x$.

EXAMPLE 3.7. Let $f = 27x^3 + 6x^2y + 12xy^2 + 81y^2 + 3x^2 + 5xy + 3y^2 + 3x + 2y + 243 \in \mathbb{Q}[x^{\pm 1}, y^{\pm 1}]$ where \mathbb{Q} has the 3-adic valuation.

Then the regular triangulation corresponding to $(\text{val}(c_{ij}))$ is shown in Figure 8, along with the tropical variety.

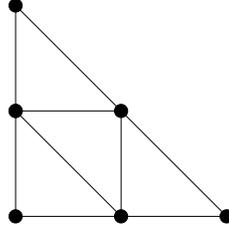
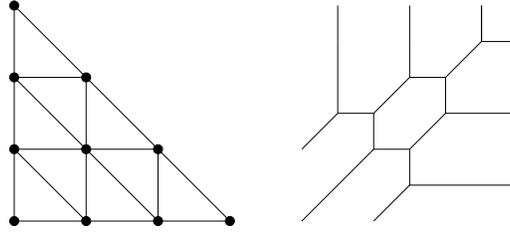
FIGURE 7. The regular subdivision induced by $(\text{val}(c_u))$ 

FIGURE 8. A tropical plane cubic

4. The Fundamental and Structure theorems

The definition of the tropical variety in Definition 2.4 was by analogy with the classical case. For an ideal I the tropical variety $\text{trop}(V(I))$ is the set of common “tropical zeros” of the tropicalizations of the polynomials $f \in I$. The connection between the tropicalization $\text{trop}(Y)$ of a variety $Y = V(I) \subseteq (K^*)^n$ and the original variety Y is closer than this analogy might suggest, as the following theorem shows.

THEOREM 4.1 (Fundamental theorem of tropical algebraic geometry). *Let K be an algebraically closed field with a nontrivial valuation $\text{val} : K^* \rightarrow \mathbb{R}$, and let Y be a subvariety of $(K^*)^n$. Then*

$$\begin{aligned} \text{trop}(Y) &= \text{cl}(\text{val}(Y)) \\ &= \text{cl}(\{(\text{val}(y_1), \dots, \text{val}(y_n)) : y = (y_1, \dots, y_n) \in Y\}), \end{aligned}$$

where the closure is in the usual Euclidean topology on \mathbb{R}^n .

EXAMPLE 4.2. Let $Y = V(x + y + 1) \subseteq (K^*)^2$, where $K = \mathbb{C}\{\{t\}\}$. Then $Y = \{(a, -1 - a) : a \in K^* \setminus \{-1\}\}$. Note that

$$(\text{val}(a), \text{val}(-1 - a)) = \begin{cases} (\text{val}(a), 0) & \text{if } \text{val}(a) > 0, \\ (\text{val}(a), \text{val}(a)) & \text{if } \text{val}(a) < 0, \\ (0, \text{val}(b)) & \text{if } a = -1 + b, \text{val}(b) > 0, \\ (0, 0) & \text{otherwise.} \end{cases}$$

Note that, as predicted by Theorem 4.1, the union of these sets is precisely $\text{trop}(Y)$, as calculated in part 2 of Example 1.2. This is illustrated in Figure 9, which should be compared with Figure 2.

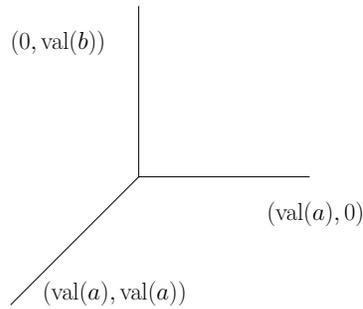


FIGURE 9. The Fundamental Theorem applied to a tropical line

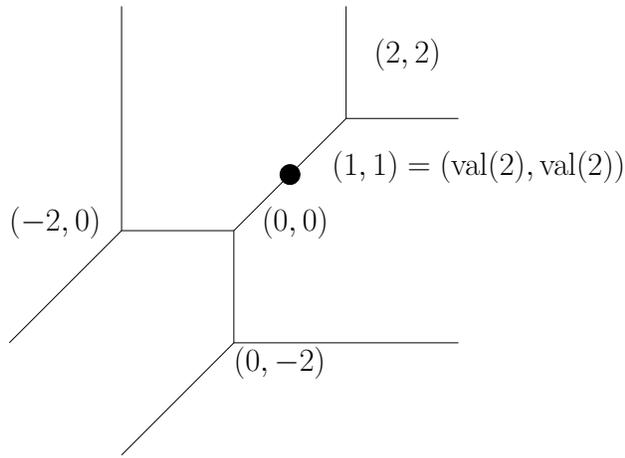


FIGURE 10. An example of the Fundamental Theorem

EXAMPLE 4.3. Let $f = 4x^2 + xy - 4y^2 + x - y - 4 \in \mathbb{Q}[x, y]$, where \mathbb{Q} has the 2-adic valuation. Then $\text{trop}(V(f))$ is shown in Figure 10. Note that the point $(2, 2) \in V(f)$, and $(\text{val}(2), \text{val}(2)) = (1, 1) \in \text{trop}(V(f))$.

Theorem 4.1 was first shown for hypersurfaces by Kapranov; see [14]. The general case first appears in work of Speyer and Sturmfels [33], [34]; see also the work of Draisma [13], Payne [32], and Jensen, Markwig, and Markwig [24]. The hard part of Theorem 4.1 is to show that if $w \in \text{trop}(Y) \cap (\text{im val})^n$ then there is $y \in Y$ with $\text{val}(y) = w$. Showing that $\{\text{val}(y) : y \in Y\} \subseteq \text{trop}(Y)$ is comparatively easy.

The slogan form of Theorem 4.1 is then:

Tropical varieties are combinatorial shadows of classical varieties.

The word “combinatorial” is justified by the Structure Theorem for tropical varieties, which gives combinatorial constraints on which sets can be tropical varieties. The statement of this theorem requires some more polyhedral definitions.

DEFINITION 4.4. Let Σ be a polyhedral complex. The *support* of Σ is the set

$$|\Sigma| = \{x \in \mathbb{R}^n : x \in \sigma \text{ for some } \sigma \in \Sigma\}.$$

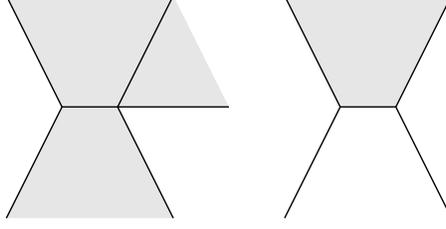


FIGURE 11. The complex on the left is pure, while the one on the right is not.

A polyhedral complex is *pure* if the dimension of every maximal polyhedron is the same. See Figure 11.

The *lineality space* L of a polyhedral complex Σ is the largest subspace of \mathbb{R}^n for which $x+l \in \Sigma$ for all $x \in \Sigma, l \in L$. In part 1 of Example 2.5 the lineality space of $\text{trop}(Y)$ is $\text{span}((1, 1, 1, 1)) \subseteq \mathbb{R}^4$.

DEFINITION 4.5. A *weighted polyhedral complex* is a pure polyhedral complex Σ with a weight $w_\sigma \in \mathbb{N}$ for all maximal-dimensional $\sigma \in \Sigma$.

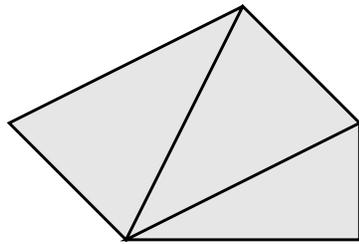
Let Σ be a weighted (im val)-rational polyhedral complex that is pure of dimension d . The complex Σ is *balanced* if the following “zero-tension” conditions hold.

- (1) If Σ is a one-dimensional rational fan, let $\mathbf{u}_1, \dots, \mathbf{u}_s$ be the first lattice points on the rays of Σ , and let w_i be the weight of the cone containing the lattice point \mathbf{u}_i . Then Σ is balanced if $\sum_{i=1}^s w_i \mathbf{u}_i = 0$.
- (2) For a general polyhedral complex Σ , fix a $(d-1)$ -dimensional polyhedron τ of Σ . Let $L = \text{span}(x-y : x, y \in \tau)$ be the affine span of τ . Let $\text{star}_\Sigma(\tau)$ be the rational polyhedral fan whose support is $\{w \in \mathbb{R}^n : \text{there exists } \epsilon > 0 \text{ for which } w' + \epsilon w \in \Sigma \text{ for all } w' \in \tau\} + L$. This has one cone for each polyhedron $\sigma \in \Sigma$ that contains τ , and has lineality space L . The quotient $\text{star}_\Sigma(\tau)/L$ is a one dimensional fan which inherits weights from Σ . We say that Σ is balanced at τ if the one-dimensional fan $\text{star}_\Sigma(\tau)/L$ is balanced. The polyhedral complex Σ is balanced if Σ is balanced at all $(d-1)$ -dimensional cones.

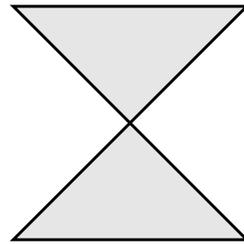
EXAMPLE 4.6. Let $f = x^2y^2 + x^3 + y^3 + 1 \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$. Then $\text{trop}(V(f))$ is a one-dimensional fan with four rays: $\text{pos}((1, 0))$, $\text{pos}((0, 1))$, $\text{pos}((-2, -1))$, and $\text{pos}((-1, -2))$. This is balanced with weights 3, 3, 1, and 1: $3(1, 0) + 3(0, 1) + 1(-2, -1) + 1(-1, -2) = (0, 0)$.

DEFINITION 4.7. We associate a graph to a pure polyhedral complex Σ , with a vertex for each maximal polyhedron $\sigma \in \Sigma$, and an edge between two vertices if the corresponding polyhedra intersect in a codimension-one face. The polyhedral complex Σ is *connected through codimension-one* if this graph is connected. For example, the polyhedral complex on the left of Figure 12 is connected through codimension-one, while the one on the right is not.

Recall that a variety $Y \subseteq (K^*)^n$ is irreducible if we cannot write $Y = Y_1 \cup Y_2$ for Y_1, Y_2 nonempty proper subvarieties of Y . Note that by Theorem 4.1 we have $\text{trop}(Y_1 \cup Y_2) = \text{trop}(Y_1) \cup \text{trop}(Y_2)$.



Connected through codimension one



Not connected through codimension one

FIGURE 12. A polyhedral complex that is (left) and is not (right) connected through codimension one

The following “Structure Theorem” summarizes the combinatorial structure on the tropical variety. This is essentially due to Bieri and Groves [3], with the stronger connectedness statement from [5] and [7].

THEOREM 4.8 (Structure Theorem). *Let Y be a d -dimensional irreducible sub-variety of $(K^*)^n$. Then $\text{trop}(Y)$ is the support of a pure d -dimensional weighted balanced (im val)-rational polyhedral complex that is connected through codimension-one.*

The Structure Theorem means that the tropical variety has a discrete structure, and records information about the original variety (such as its dimension).

The weights w_σ on maximal cones that make $\text{trop}(Y)$ balanced can be computed from Y , as we explain in the next section.

5. The computational approach to tropical varieties

An important aspect of tropical varieties is that they can actually be computed in practice. This uses an extension of the theory of Gröbner bases to fields with a valuation.

Fix a splitting $(\text{im val}) \rightarrow K^*$ of the valuation. This is a group homomorphism $u \mapsto t^u$ with $\text{val}(t^u) = u$. For example, when $K = \mathbb{C}$ with the trivial valuation ($\text{val}(a) = 0$ for all $a \neq 0$), then we can choose the splitting $0 \mapsto 1$. When $K = \mathbb{C}\{\{t\}\}$, we can choose $u \mapsto t^u$, and when $K = \mathbb{Q}$ with the p -adic valuation, we can choose $u \mapsto p^u$. Such splittings always exist when K is algebraically closed; see [28, Lemma 2.1.13].

Let $R = \{a \in K : \text{val}(a) \geq 0\}$ be the valuation ring of K . The ring R is local, with maximal ideal $\mathfrak{m} = \{a \in K : \text{val}(a) > 0\} \cup \{0\}$. The quotient $\mathbb{k} = R/\mathfrak{m}$ is the residue field.

- EXAMPLE 5.1.**
- (1) When $K = \mathbb{C}$ has the trivial valuation, we have $R = \mathbb{C}$, and $\mathfrak{m} = 0$, so $\mathbb{k} = \mathbb{C}$.
 - (2) When $K = \mathbb{C}\{\{t\}\}$, $R = \bigcup_{n \geq 1} \mathbb{C}\llbracket t^{1/n} \rrbracket$, and $\mathbb{k} = \mathbb{C}$.
 - (3) When $K = \mathbb{Q}_p$, $R = \mathbb{Z}_p$, and $\mathbb{k} = \mathbb{Z}/p\mathbb{Z}$.

Given a polynomial $f = \sum c_u x^u \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and $w \in (\text{im val})^n$, the *initial form* is

$$\text{in}_w(f) = \sum_{\text{val}(c_u) + w \cdot u = \text{trop}(f)(w)} \overline{t^{-\text{val}(c_u)} c_u} x^u \in \mathbb{k}[x_1^{\pm 1}, \dots, x_n^{\pm 1}],$$

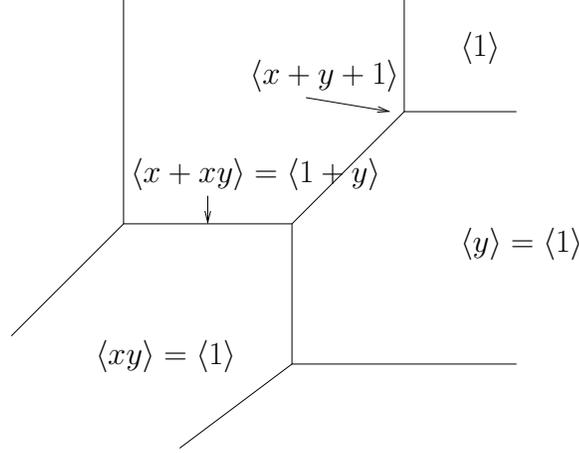


FIGURE 13. The Gröbner complex of Example 5.4

where for $a \in R$ we denote by \bar{a} the image of a in \mathbb{k} .

EXAMPLE 5.2. Let $K = \mathbb{Q}$ with the 2-adic valuation, and let $f = 2x^2 + xy + 6y^2 + 5x - 3y + 4$. Then for $w = (2, 2)$ we have $\text{trop}(f)(w) = 2$, so $\text{in}_w(f) = \bar{5}x + \bar{-3}y + \bar{2^{-2}}4 = x + y + 1 \in \mathbb{Z}/2\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$.

For $w = (-2, -1)$ we have $\text{trop}(f)(w) = -3$, so $\text{in}_w(f) = x^2 + xy$.

Given an ideal $I \subset K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and $w \in (\text{im val})^n$, the initial ideal of I is

$$\text{in}_w(I) = \langle \text{in}_w(f) : f \in I \rangle.$$

This is a variant of the standard theory of Gröbner bases for ideals in a polynomial ring; see [8] for an excellent introduction to the standard case. As in that setting, the initial ideal need not be generated by the initial forms of a generating set for I . There are always finite generating sets for I , however, for which this is the case. These finite sets (Gröbner bases) can be computed using a variant of the standard Gröbner basis algorithm.

The connection of this theory of Gröbner bases to tropical geometry is the following computational characterization of tropical varieties, due to Sturmfels [36, Chapter 9].

PROPOSITION 5.3. *Let $Y = V(I) \subseteq (K^*)^n$ and let $w \in (\text{im val})^n$. Then $w \in \text{trop}(Y)$ if and only if $\text{in}_w(I) \neq \langle 1 \rangle \subseteq \mathbb{k}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.*

EXAMPLE 5.4. Let $I = \langle 2x^2 + xy + 6y^2 + 5x - 3y + 4 \rangle \in \mathbb{Q}[x^{\pm 1}, y^{\pm 1}]$, where \mathbb{Q} has the 2-adic valuation. The claim of Proposition 5.3 is illustrated in Figure 13.

There is a polyhedral complex Σ with $\text{in}_w(I)$ constant for $w \in \text{relint}(\sigma)$ for any $\sigma \in \Sigma$. This is the *Gröbner complex* of a homogenization of I . In the case that $K = \mathbb{C}$ with the trivial valuation the Gröbner complex is a polyhedral fan, called the Gröbner fan. See [2], [31], [35, Chapter 2] for more on the Gröbner fan, and [27], [28] for expositions of the Gröbner complex.

Proposition 5.3 implies that $\text{trop}(Y)$ is the union of the polyhedra σ in the Gröbner complex of $I(Y)$ for which $\text{in}_w(I) \neq \langle 1 \rangle$ for any $w \in \text{relint}(\sigma)$. The

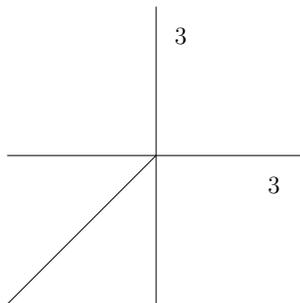


FIGURE 14. The balancing condition

software `gfan` [23] by Anders Jensen computes tropical varieties by exploiting this Gröbner description.

The Structure Theorem (Theorem 4.8) asserts that $\text{trop}(Y)$ is the support of a weighted balanced polyhedral complex. The weights that make the polyhedral complex balanced can be defined using the Gröbner theory we have just summarized. We assume that the polyhedral complex structure Σ has been chosen, as described above, so that $\text{in}_w(I(Y))$ is constant on the relative interior of polyhedra of Σ . Fix $w \in (\text{im val})^n$ in the relative interior of a maximal polyhedron of Σ . Then $V(\text{in}_w(I(Y))) \subseteq (\mathbb{k}^*)^n$ is a union of $(\mathbb{k}^*)^d$ -orbits, where $d = \dim(Y)$ (see [28, Chapter 3] for details). We set w_σ to be the number of such orbits (counted with multiplicity).

Hidden in the proof of the Structure Theorem is the fact that this choice makes the polyhedral complex Σ balanced.

EXAMPLE 5.5. Let $f = x^2 + 3x + 2 + x^2y + 2xy^2 - 2y^2 \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$. Then $\text{trop}(V(f))$ is a one-dimensional fan with five rays, spanned by the vectors $\{(1, 0), (0, 1), (-1, 0), (-1, -1), (0, -1)\}$. When $w = (0, 1)$, $\text{trop}(f)(w) = 0$, so $\text{in}_w(f) = x^2 + 3x + 2 = (x + 2)(x + 1)$. We then have $V(\text{in}_w(f)) = \{(-2, a) : a \in \mathbb{C}^*\} \cup \{(-1, a) : a \in \mathbb{C}^*\}$. The weight on the cone spanned by $(0, 1)$ is thus 2. When $w = (-1, -1)$, $\text{in}_w(f) = x^2y + 2xy^2$, so $V(\text{in}_w(f)) = V(x + 2y) = \{(2a, -a) : a \in \mathbb{C}^*\}$. This makes the weight on this cone 1. Similarly, the weight on the cone spanned by $(1, 0)$ is 2 and all other weights are 1. Note that

$$2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ -1 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is illustrated in Figure 14.

6. Connections to subvarieties of toric varieties

Tropical varieties record information not just about subvarieties of the algebraic torus $(K^*)^n$, but also about their compactifications. This vastly extends the reach of tropical techniques; for example, every projective variety is a compactification of a subvariety of the algebraic torus. The connection is through the theory of toric varieties.

We assume here that $K = \mathbb{C}$. This means that for $Y \in (\mathbb{C}^*)^n$, $\text{trop}(Y)$ is the support of a weighted balanced rational polyhedral fan.

DEFINITION 6.1. A (normal) *toric variety* is a normal variety X containing a dense copy of $T = (\mathbb{C}^*)^n$ with an action of T on X that extends the action of T on itself.

Examples include:

- (1) $X = (\mathbb{C}^*)^n$,
- (2) $X = \mathbb{A}^n \supset (\mathbb{C}^*)^n = \{x \in \mathbb{A}^n : x_i \neq 0 \text{ for } 1 \leq i \leq n\}$,
- (3) $X = \mathbb{P}^n \supset (\mathbb{C}^*)^n = \{x \in \mathbb{P}^n : x_i \neq 0 \text{ for } 0 \leq i \leq n\}$,
- (4) $X = \mathbb{P}^1 \times \mathbb{P}^1 \supset (\mathbb{C}^*)^2$.

A toric variety X is a union of T -orbits. These can be recorded using a polyhedral fan Σ .

EXAMPLE 6.2. The projective plane \mathbb{P}^2 decomposes into the following $T = (\mathbb{C}^*)^2$ -orbits:

$$(\mathbb{C}^*)^2 \cup \{[0 : a : b] : a, b \in \mathbb{C}^*\} \cup \{[a : 0 : b] : a, b \in \mathbb{C}^*\} \cup \{[a : b : 0] : a, b \in \mathbb{C}^*\} \\ \cup \{[1 : 0 : 0]\} \cup \{[0 : 1 : 0]\} \cup \{[0 : 0 : 1]\}.$$

The corresponding fan is shown in Figure 15.

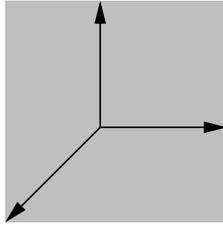


FIGURE 15. The fan of \mathbb{P}^2

Alternatively (and more standardly), given a rational polyhedral fan Σ we construct a toric variety X_Σ by gluing together torus orbits closures. Each cone of Σ determines an affine toric variety, and the fan tells us how to glue them together. For example, for \mathbb{P}^2 the fan tells us to construct \mathbb{P}^2 by gluing together the three affine charts $\{x \in \mathbb{P}^2 : x_i \neq 0\}$ for $0 \leq i \leq 2$. For more background on toric varieties, some good references include [9] and [17].

The connection to tropical geometry begins with the following question.

QUESTION 6.3. Given a toric variety X_Σ , and a subvariety $\overline{Y} \subseteq X_\Sigma$, which T -orbits of X_Σ does \overline{Y} intersect?

Surprisingly, the answer uses tropical geometry. The subvariety \overline{Y} intersects the torus orbit indexed by $\sigma \in \Sigma$ if and only if $\text{trop}(\overline{Y} \cap T)$ intersects $\text{relint}(\sigma)$. This follows from work of Tevelev [38].

EXAMPLE 6.4. Let $\overline{Y} = V(x+y+z) \subseteq \mathbb{P}^2$. Then $\text{trop}(\overline{Y} \cap T) = \text{trop}(x+y+1)$, which is the standard tropical line. This intersects every one of the cones of the fan of \mathbb{P}^2 except for the top-dimensional ones. Indeed, the top-dimensional cones correspond to the T -fixed points $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$, which do not lie in \overline{Y} , while every other T -orbit does contain a point of \overline{Y} .

A fundamental problem in algebraic geometry, especially in the consideration of moduli spaces, is to find a good compactification of a variety. Here the notion of “good” will depend on the specific problem, but we often require the compactification to be smooth, the “boundary” (new points added in the compactification) to be a divisor (codimension-one), and the irreducible components of this divisor to intersect nicely.

EXAMPLE 6.5. Let $\mathcal{A} = \{H_1, \dots, H_s\}$ be a hyperplane arrangement in \mathbb{P}^n , where $H_i = \{x \in \mathbb{P}^n : a_i \cdot x = 0\}$, with $a_i \in \mathbb{C}^{n+1}$. Let $Y = \mathbb{P}^n \setminus \mathcal{A}$. This can be embedded into $(\mathbb{C}^*)^{s-1}$ by sending $y \in Y$ to $[a_1 \cdot y : \dots : a_s \cdot y]$. Let A be the $(n+1) \times s$ matrix with columns the vectors a_i , which we assume has rank $n+1$, and let B be a $(s-n-1) \times (n+1)$ matrix of rank $s-n-1$ with $AB^T = 0$. Then $Y = V(\sum_{i=1}^n b_{ij}x_j : 1 \leq i \leq s-n-1)$. One choice of compactification of Y is the original \mathbb{P}^n ; another is the DeConcini-Procesi wonderful compactification [10].

A natural way to compactify a variety is to take its closure in a larger variety. The idea we follow, due to Tevelev [38], is to take as the larger variety a toric variety whose fan has support the tropical variety. Note that if $\dim(Y) > 1$ there is not a unique such fan, and there may be no preferred choice.

DEFINITION 6.6. Fix $Y \subset T$, and choose a fan Σ with support $\text{trop}(Y)$. The closure $\overline{Y} = \text{cl}(Y \subset X_\Sigma)$ is a *tropical compactification* of Y .

Tropical compactifications have nice properties:

- (1) \overline{Y} is proper (compact),
- (2) \overline{Y} intersects a codimension- k T -orbit of X_Σ in codimension k .

The first condition is somewhat surprising, since the toric variety X_Σ is not itself compact. For example, if $Y = V(x+y+1) \subseteq (\mathbb{C}^*)^2$, then $\text{trop}(Y)$ is the standard tropical line. There is only one polyhedral fan Σ with this support, and the corresponding toric variety is \mathbb{P}^2 with three points removed, which is not compact. However in this case the closure of Y in \mathbb{P}^2 , which is clearly compact, does not pass through these three points, so the closure of Y in X_Σ equals the closure of Y in \mathbb{P}^2 , so is also compact.

The second condition means that the pull-backs to \overline{Y} of the codimension- k torus-invariant strata on X_Σ give a finite number of distinguished classes in the Chow group $A^k(\overline{Y})$. For example when $k=1$ this gives a distinguished collection of effective divisors on \overline{Y} .

If the fan Σ is chosen to be sufficiently refined then we get further control over the compactification \overline{Y} . One way to guarantee “sufficiently refined” is to choose a fan Σ so that $\text{in}_w(I(Y))$ is constant for all $w \in \text{relint } \sigma$ for all $\sigma \in \Sigma$. By further refining Σ we can also assume that the toric variety X_Σ is smooth. With a sufficiently refined fan we have:

- (1) The multiplicity w_σ on a maximal cone σ of Σ equals the intersection number $[\overline{Y}] \cdot [V(\sigma)]$, where $V(\sigma)$ is the closure of the T -orbit corresponding to σ .
- (2) For any $\sigma \in \Sigma$, the intersection of \overline{Y} with the T -orbit \mathcal{O}_σ is the quotient $V(\text{in}_w(I(Y)))/(\mathbb{C}^*)^{\dim \sigma}$ for any $w \in \text{relint}(\sigma)$.

See also [22], [33], and [37] for more details on this. The connection to intersection theory has been further developed by Allermann and Rau [1], which is the tropical analogue of earlier toric work by Fulton and Sturmfels [18].

EXAMPLE 6.7. When $Y = \mathbb{P}^n \setminus \mathcal{A}$, as in Example 6.5, there is a coarsest fan structure Σ on $\text{trop}(Y) \subseteq \mathbb{R}^{s-1}$, since every other fan with support $\text{trop}(Y)$ has cones that subdivide those of Σ . The tropical compactification \overline{Y} coming from taking the closure in X_Σ is the DeConcini/Procesi wonderful compactification for most choices of \mathcal{A} ; see [15] and [38].

A motivating example of this is given by the moduli space $\overline{M}_{0,n}$. The moduli space $M_{0,n}$ parameterizes isomorphism classes of smooth genus zero curves with n distinct marked points. It thus parameterizes ways to arrange n distinct point on \mathbb{P}^1 up to $\text{Aut}(\mathbb{P}^1)$. For example, $M_{0,3}$ is a point, as there is an automorphism of \mathbb{P}^1 that takes any three distinct points to $0, 1, \infty$. When $n = 4$, $M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. In general,

$$\begin{aligned} M_{0,n} &= (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \text{diagonals} \\ &= (\mathbb{C}^* \setminus \{1\})^{n-3} \setminus \text{diagonals} \\ &= \mathbb{P}^{n-3} \setminus \{x_0 = 0, x_i = 0, x_i = x_0, x_i = x_j : 1 \leq i < j \leq n\}. \end{aligned}$$

We thus have $M_{0,n}$ as the complement of $\binom{n-1}{2} = \binom{n}{2} - n + 1$ hyperplanes. This means the moduli space $M_{0,n}$ can be embedded into $(\mathbb{C}^*)^{\binom{n}{2}-n}$ as a closed subvariety as in Example 6.5. The tropical variety $\text{trop}(M_{0,n})$ is the support of a $(n-3)$ -dimensional fan Δ in $\mathbb{R}^{\binom{n}{2}-n}$. The toric variety X_Δ is smooth, but not complete (projective).

The fan Δ is the space of phylogenetic trees, which comes from mathematical biology. See, for example, [4] for more on this space, and [34] for some of the connection, which uses Kapranov's description [25] of $\overline{M}_{0,n}$ as the Chow quotient of a Grassmannian. Maximal cones of Δ are labelled by trivalent trees with n labelled leaves. A point in the cone records the length of the internal edges in the tree.

A picture of Δ when $n = 5$ is shown in Figure 16. This is a two-dimensional fan in \mathbb{R}^5 , so its intersection with the four-dimensional sphere in \mathbb{R}^5 is a graph, which is drawn in Figure 16.

The closure of $M_{0,n}$ in X_Δ is the Deligne-Mumford moduli space $\overline{M}_{0,n}$ of stable genus zero curves with n marked points. This parameterizes isomorphism classes of trees of \mathbb{P}^1 's with n marked points and at least three special points (nodes or marked points) on each component. See [21] or [38] for details.

For $\sigma \in \Delta$, the intersection of $\overline{M}_{0,n}$ with the torus orbit corresponding to σ is the stratum of all curves with dual graph the corresponding tree. In particular, the intersection of $\overline{M}_{0,n}$ with a torus-invariant divisor on X_Δ is a boundary divisor.

The moduli space $\overline{M}_{0,n}$ and the toric variety X_Δ are closely related. Their Picard groups are isomorphic, and the inclusion $i : \overline{M}_{0,n} \rightarrow X_\Delta$ introduces an isomorphism $i^* : A^*(X_\Delta) \rightarrow A^*(\overline{M}_{0,n})$. This generalizes to general wonderful compactifications; see [16].

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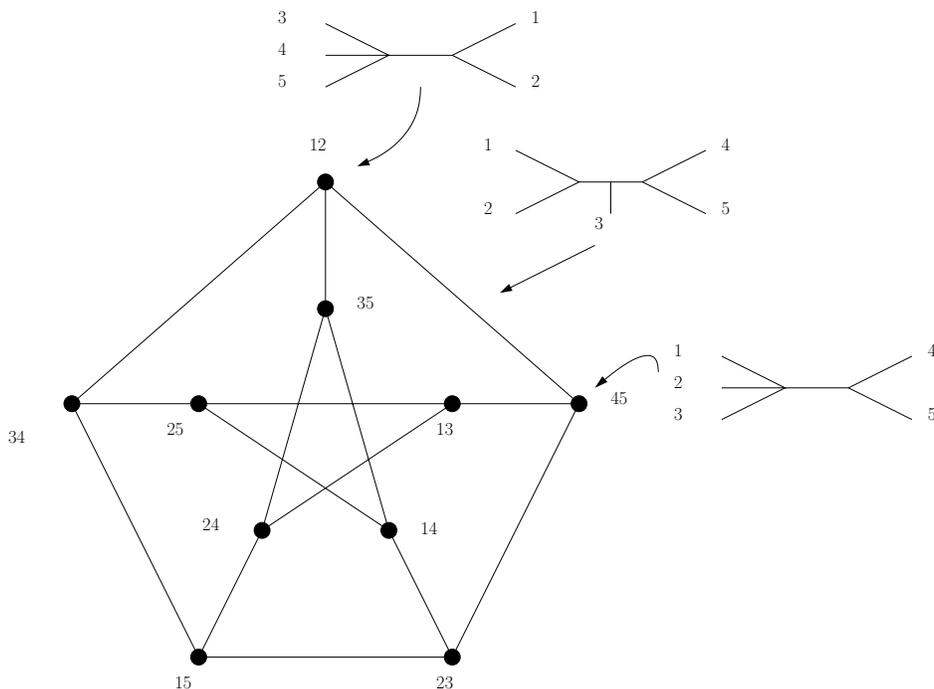


FIGURE 16. The tropical variety of $M_{0,5}$

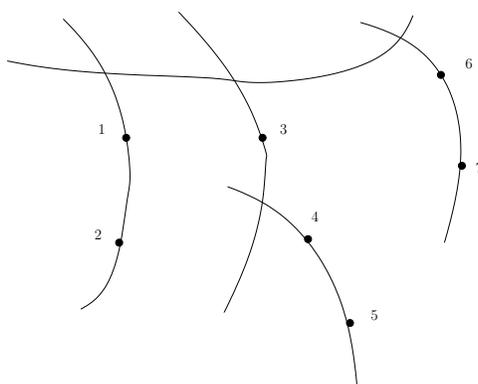


FIGURE 17. A stable curve with 7 marked points

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